

Phys 637, I-Semester 2022/23, Assignment 4

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Due-date: Lecture 14.10.2022

(1) Derivation of Born-Markov Master equation: Work out some of the missing intermediate steps of the derivation of the Born-Markov Master equation given in the lecture:

(1a) Show that the total interaction picture density matrix obeys the Liouville-von Neumann equation [(4.8) of lecture] [3 points].

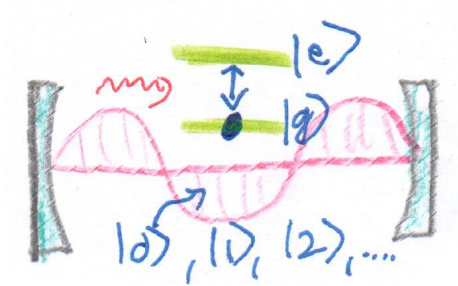
$$\frac{d}{dt}\hat{\rho}^{(I)}(t) = -i[\hat{H}_{\text{int}}^{(I)}(t), \hat{\rho}^{(I)}(t)], \quad (1)$$

(1b) Show that for interaction picture density matrices, we have $\hat{\rho}_S^{(I)}(t) = \text{Tr}_E\{\hat{\rho}^{(I)}(t)\}$ [above (4.11) of lecture]. [2 points]

(1c) Assume the first term of Eq. (4.11) of the lecture is nonzero for a certain splitting of \hat{H}_0 and H_{int} . By construction show how \hat{H}_0 and H_{int} can be re-arranged to make the term vanish, without changing essential physics. [3 points]

(1d) Find a counter-example demonstrating that in general the partial trace is not cyclic [yellow box after 4.17 of lecture]. [2 points]

(2) Two-level atom in an optical cavity, Kraus operators: Consider an atom with only two relevant electronic states $|g\rangle$ and $|e\rangle$, resonantly coupled to the mode of light in an optical cavity as shown below.



We denote a quantum state with exactly n -photons in the cavity with $|n\rangle$, and handle it like harmonic oscillator states. In the dipole and rotating wave approximation¹ the Hamiltonian defined what is called the “Jaynes-Cummings-model”:

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\Delta E}{2}\hat{\sigma}_z + \frac{\hbar\kappa}{2}(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-), \quad (2)$$

where $\hat{\sigma}_\pm = \sigma_x \pm i\sigma_y$. We want to treat the atom as system and single photon mode as environment, even though a single photon mode usually is not complex enough to be

¹You do not yet need to know what these are, see PHY402.

called “environment”.

(2a) Our complete basis is $\{|g, n\rangle, |e, n\rangle$ for integer n . Show that in terms of this basis the Hamiltonian decomposes into 2 by 2 blocks and find those blocks explicitly in matrix form. If the initial state was $|e\rangle \otimes |0\rangle$, list all basis states that can become occupied in time. [2 points]

(2b) Find the time evolution operator, and from that the evolution of the reduced density matrix for the atom from the initial state $|e\rangle \otimes |0\rangle$. [4 points]

(2c) Now we want to use this example to understand Kraus operators. From (b) and the initial state $|e\rangle \otimes |0\rangle$, find the Kraus operators of the problem. Then use those to find the reduced density matrix of the atom and compare with your result from (b). [4 points]

(2d) (bonus) How do the Kraus operators change if we had a more complex initial state $|e\rangle \otimes \sum_n c_n |n\rangle$, with multiple non-zero entries. What does this imply for the system evolution?

(3) Numerical solution of Jaynes-Cummings model: We now want to verify results from Q2 numerically, and also setup a tool that can solve the dynamics for more general cases.

(3a) Using the expression

$$|\psi(t)\rangle = \sum_{a \in \{g, e\}, n} c_{an}(t) |e, n\rangle, \quad (3)$$

for the general time-dependent state expressed in the basis advocated in Q2, derive equations of motion for the coefficients $c_{an}(t)$ and implement those in the template `Assignment4_code_draft_v1.xmms`. Run this without changing parameters and as usual first check whether your equations conserve energy and normalisation of the state, using `Assignment4_plot_checks_v1.m`. [5 points]

(3b) The code is set up to sample the reduced density matrix of the atom in two different way, once directly from the time evolving state (you can rederive this for practice if you like) and once via the Kraus operators. You still have to insert your result for the Kraus operator from Q2 into the code at the indicated position. Also derive how a 2 by 2 Kraus operator acts on a 2 by 2 density matrix (give an explicit equation) and verify the construction in the block around line 198. Finally use the script `Assignment4_compare_reddm_v1.m` to compare both approaches (verifying your Kraus operators from Q2). Discuss why the reduced DM evolves in the way it does. [5 points]

(3c) (bonus) Using the more advanced code `Assignment4_advancedcode_draft_v1.xmms` (provided later, remind me otherwise), also check your answer to (2d) and discuss.