Phys 637, I-Semester 2022/23, Assignment 4 solution

(1) Derivation of Born-Markov Master equation:

Work out some of the missing intermediate steps of the derivation of the Born-Markov Master equation given in the lecture:
(1a) Show that the total interaction picture density matrix obeys the Liouville-von Neumann equation [(4.8) of lecture] [5 points].

$$\frac{d}{dt}\hat{\rho}^{(I)}(t) = -i\big[\hat{H}^{(I)}_{\rm int}(t), \hat{\rho}^{(I)}(t)\big], \tag{1}$$

Density operator in interaction picture ($\hbar = 1$)

$$\hat{\rho}^{(I)}(t) = e^{i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t}$$

Taking time derivative,

$$\frac{d}{dt}\hat{\rho}^{(I)}(t) = ie^{i\hat{H}_0 t}\big[\hat{H}_0, \hat{\rho}(t)\big]e^{-i\hat{H}_0 t} + e^{i\hat{H}_0 t}\frac{d}{dt}\hat{\rho}(t)e^{-i\hat{H}_0 t}$$

use von Neumann equation (3.2),

$$\frac{d}{dt}\hat{\rho}^{(I)}(t) = ie^{i\hat{H}_0 t}[\hat{H}_0, \hat{\rho}(t)]e^{-i\hat{H}_0 t} - ie^{i\hat{H}_0 t}[\hat{H}(t), \hat{\rho}(t)]e^{-i\hat{H}_0 t}$$

putting, $\hat{H}(t) = \hat{H}_0 + \hat{H}_{int}$

$$\frac{d}{dt}\hat{\rho}^{(I)}(t) = -ie^{i\hat{H}_0 t}[\hat{H}_{int}(t), \hat{\rho}(t)]e^{-i\hat{H}_0 t}$$

$$= -ie^{i\hat{H}_0 t}\hat{H}_{int}(t)\hat{\rho}(t)e^{-i\hat{H}_0 t} + ie^{i\hat{H}_0 t}\hat{H}_{int}(t)\hat{\rho}(t)e^{-i\hat{H}_0 t}$$

$$= -ie^{i\hat{H}_0 t}\hat{H}_{int}(t)e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t} + ie^{i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}\hat{H}_{int}(t)e^{-i\hat{H}_0 t}$$

$$= -i\hat{H}^{(I)}_{int}(t)\hat{\rho}^{(I)}(t) + i\hat{\rho}^{(I)}(t)\hat{H}^{(I)}_{int}(t)$$

$$= -i\big[\hat{H}^{(I)}_{int}(t), \hat{\rho}^{(I)}(t)\big]$$

(1b) Show that for interaction picture density matrices, we have $\hat{\rho}^{(I)}_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{E}}\big\{\hat{\rho}^{(I)}(t)\big\}$ [above (4.10) of lecture]. [3 points]

we have,

$$Tr_{\mathcal{E}}\big\{\hat{\rho}^{(I)}(t)\big\} = Tr_{\mathcal{E}}\big\{e^{i(\hat{H}_s+\hat{H}_{\mathcal{E}})t}\hat{\rho}(t)e^{-i(\hat{H}_s+\hat{H}_{\mathcal{E}})t}\big\}$$

$$= e^{i\hat{H}_s t}\, Tr_{\mathcal{E}}\big\{e^{i\hat{H}_{\mathcal{E}} t}\hat{\rho}(t)e^{-i\hat{H}_{\mathcal{E}} t}\big\}e^{-i\hat{H}_s t} \tag{2}$$

$$= e^{i\hat{H}_s t}\, Tr_{\mathcal{E}}\big\{\hat{\rho}(t)\big\}e^{-i\hat{H}_s t}$$

$$= \hat{\rho}_{\mathcal{S}}^{(I)}(t)$$

Where in the second line we have used the invariance of the trace operation under cyclic permutations of the arguments (see 1d for proof).

(1c) Assume the first term of Eq. (4.11) of the lecture is nonzero for a certain splitting of $\hat{H}_0$ and $\hat{H}_{\rm int}$. By construction show how $\hat{H}_0$ and $\hat{H}_{\rm int}$ can be re-arranged to make the term vanish, without changing essential physics. [5 points]

The first term is,

$$-i\, Tr_{\mathcal{E}}\big[\hat{H}_{int}^{(I)}(t), \hat{\rho}(0)\big] = -i\, Tr_{\mathcal{E}}\big[\hat{H}_{int}^{(I)}(t), \hat{\rho}_s\hat{\rho}_{\mathcal{E}}\big]$$

$$= -i\, Tr_{\mathcal{E}}\big\{\hat{H}_{int}^{(I)}(t)\hat{\rho}_s\hat{\rho}_{\mathcal{E}} - \hat{\rho}_s\hat{\rho}_{\mathcal{E}}\hat{H}_{int}^{(I)}(t)\big\}$$

$$= -i\, Tr_{\mathcal{E}}\big\{\hat{H}_{int}^{(I)}(t)\hat{\rho}_s\hat{\rho}_{\mathcal{E}}\big\} + i\, Tr_{\mathcal{E}}\big\{\hat{\rho}_s\hat{\rho}_{\mathcal{E}}\hat{H}_{int}^{(I)}(t)\big\}$$

$$= -i\, Tr_{\mathcal{E}}\big\{\hat{H}_{int}^{(I)}(t)\hat{\rho}_{\mathcal{E}}\big\}\hat{\rho}_s + i\hat{\rho}_s\, Tr_{\mathcal{E}}\big\{\hat{\rho}_{\mathcal{E}}\hat{H}_{int}^{(I)}(t)\big\}$$

$$= -i\big[\, Tr_{\mathcal{E}}\big(\hat{H}_{int}^{(I)}(t)\hat{\rho}_{\mathcal{E}}\big), \hat{\rho}_s\big] \tag{3}$$

Let,

$$\hat{H}'_{int} = \hat{H}_{int} - \, Tr_{\mathcal{E}}\big\{\hat{H}_{int}\hat{\rho}_{\mathcal{E}}\big\} \tag{4}$$

$$\hat{H}'_0 = \hat{H}_0 + \, Tr_{\mathcal{E}}\big\{\hat{H}_{int}\hat{\rho}_{\mathcal{E}}\big\} \tag{5}$$

Such that the total Hamiltonian remains unchanged. Now we can write,

$$Tr_{\mathcal{E}}\big\{\hat{H}_{int}^{\prime(I)}(t)\hat{\rho}_{\mathcal{E}}\big\} = \, Tr_{\mathcal{E}}\big\{e^{i\hat{H}'_0 t}\hat{H}'_{int}e^{-i\hat{H}'_0 t}\hat{\rho}_{\mathcal{E}}\big\}$$

putting $H'_0 = H'_s + H'_{\mathcal{E}}$ and H'_{int} from equation (4), we get

$$Tr_{\mathcal{E}}\big\{\hat{H}_{int}^{\prime(I)}(t)\hat{\rho}_{\mathcal{E}}\big\} = \, Tr_{\mathcal{E}}\big\{e^{i\hat{H}'_0 t}\hat{H}_{int}e^{-i\hat{H}'_0 t}\hat{\rho}_{\mathcal{E}}\big\} - \, Tr_{\mathcal{E}}\big\{e^{i\hat{H}'_0 t}\underbrace{Tr_{\mathcal{E}}\big(\hat{H}_{int}\hat{\rho}_{\mathcal{E}}\big)}_{\hat{O}_S}e^{-i\hat{H}'_0 t}\hat{\rho}_{\mathcal{E}}\big\}$$

$$= e^{iH'_S t}\, Tr_{\mathcal{E}}\big\{e^{i\hat{H}'_{\mathcal{E}} t}\hat{H}_{int}e^{-i\hat{H}'_{\mathcal{E}} t}\hat{\rho}_{\mathcal{E}}\big\}e^{-iH'_S t}$$

$$-e^{iH'_S t}\underbrace{Tr_{\mathcal{E}}\big\{\hat{H}_{int}\hat{\rho}_{\mathcal{E}}\big\}}_{\hat{\mathcal{O}}_S}e^{-iH'_S t}\,Tr_{\mathcal{E}}\big\{e^{i\hat{H}'_{\mathcal{E}} t}e^{-i\hat{H}'_{\mathcal{E}} t}\hat{\rho}_{\mathcal{E}}\big\}$$

$$= e^{iH'_S t}\,Tr_{\mathcal{E}}\big\{\hat{H}_{int}\hat{\rho}_{\mathcal{E}}\big\}e^{-iH'_S t} - e^{iH'_S t}\,Tr_{\mathcal{E}}\big\{\hat{H}_{int}\hat{\rho}_{\mathcal{E}}\big\}e^{-iH'_S t}$$

$$= 0,$$

Now from equation (3), we have

$$-i\,Tr_{\mathcal{E}}\big[\hat{H}'^{(I)}_{int}(t),\hat{\rho}(0)\big] = -i\big[Tr_{\mathcal{E}}\big\{\hat{H}'^{(I)}_{int}(t)\hat{\rho}_{\mathcal{E}}\big\},\hat{\rho}_s\big] = 0$$

In the second line, we have used the fact that $\hat{\mathcal{O}}_S$ is an operator that acts only on the system and hence can be brought out of the $Tr_{\mathcal{E}}$.

(1d) Find a counter-example demonstrating that in general the partial trace is not cyclic [yellow box after 4.17 of lecture]. [2 points]

We calculate,

$$Tr_{\mathcal{E}}\{\hat{S}_\alpha\hat{E}_\alpha\hat{\rho}\} - Tr_{\mathcal{E}}\{\hat{E}_\alpha\hat{S}_\alpha\hat{\rho}\}$$

$$= Tr_{\mathcal{E}}\{\hat{S}_\alpha\hat{E}_\alpha\hat{\rho} - \hat{E}_\alpha\hat{S}_\alpha\hat{\rho}\}$$

$$= Tr_{\mathcal{E}}\big[\hat{S}_\alpha\hat{E}_\alpha,\hat{\rho}\big]$$

But $\hat{S}_\alpha\hat{E}_\alpha$ do not commute with $\hat{\rho}$ so,

$$\mathrm{Tr}_{\mathcal{E}}\{\hat{S}_\alpha\hat{E}_\alpha\hat{\rho}\} \neq \mathrm{Tr}_{\mathcal{E}}\{\hat{E}_\alpha\hat{S}_\alpha\hat{\rho}\}$$

But it is not always true, partial trace can be cyclic, see the following example.

Let a bipartite Hilbert space $H_S \otimes H_{\mathcal{E}}$, ***the partial trace is cyclic when the joint operator acts as identity on the non traced subspaces****, i.e.*

$$Tr_{\mathcal{E}}\{\hat{\rho}(\hat{I}_S\otimes\hat{E})\} = Tr_{\mathcal{E}}\{(\hat{I}_S\otimes\hat{E})\hat{\rho}\}$$

where $\hat{\rho}$ is any operator acting on the joint Hilbert space, $\hat{E}$ is the operator acting on H_E and $\hat{I}_S$ is the identity operator acting on H_S.

Proof:

To prove this we use an explicit representation of the $\hat{\rho}$ operator($|\, s_k\,\rangle$ is a basis of H_S and $|\, e_m\,\rangle$ of $H_{\mathcal{E}}$)

$$\hat{\rho} = \sum_{klmn}\rho_{kl,mn}|\, s_k\,\rangle\langle\, s_l\,|\otimes|\, e_m\,\rangle\langle\, e_n\,|$$

so that,

$$Tr_{\mathcal{E}}\{\hat{\rho}(\hat{I}_S \otimes \hat{E})\} = \sum_j \sum_{klmn} \rho_{kl,mn} |\, s_k \,\rangle\langle\, s_l \,| \{ \langle\, e_j \,|\, e_m \,\rangle\langle\, e_n \,|\hat{E}|\, e_j \,\rangle \}$$

now introducing the identity $\hat{I}_{\mathcal{E}} = \sum_r |\, e_r \,\rangle\langle\, e_r \,|$ *right before* $\hat{E}$*, we get*

$$Tr_{\mathcal{E}}\{\hat{\rho}(\hat{I}_S \otimes \hat{E})\} = \sum_{rj} \sum_{klmn} \rho_{kl,mn} |\, s_k \,\rangle\langle\, s_l \,| \{ \langle\, e_j \,|\, e_m \,\rangle\langle\, e_n \,|\, e_r \,\rangle\langle\, e_r \,|\hat{E}|\, e_j \,\rangle \}$$

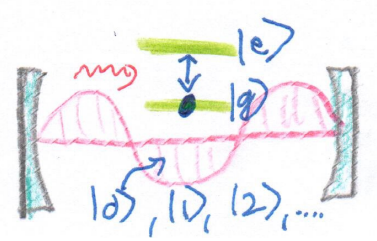
on rearranging,

$$Tr_{\mathcal{E}}\{\hat{\rho}(\hat{I}_S \otimes \hat{E})\} = \sum_r \langle\, e_r \,|(\hat{I}_S \otimes \hat{E}) \{ \sum_{klmn} \rho_{kl,mn} |\, s_k \,\rangle\langle\, s_l \,| \otimes |\, e_m \,\rangle\langle\, e_n \,| \} |\, e_r \,\rangle$$

$$= Tr_{\mathcal{E}}\{(\hat{I}_S \otimes \hat{E})\hat{\rho}\}$$

The same procedure can be used to prove that the partial trace in equation (2) is cyclic.

(2) Two-level atom in an optical cavity, Kraus operators: Consider an atom with only two relevant electronic states $|\, g \,\rangle$ and $|\, e \,\rangle$, resonantly coupled to the mode of light in an optical cavity as shown below.



We denote a quantum state with exactly n-photons in the cavity with $|\, n \,\rangle$, and handle it like harmonic oscillator states. In the dipole and rotating wave approximation[1] the Hamiltonian defined what is called the "Jaynes-Cummings-model":

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \frac{\Delta E}{2}\hat{\sigma}_z + \frac{\hbar\kappa}{2}(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-), \tag{6}$$

where $\hat{\sigma}_\pm = \sigma_x \pm i\sigma_y$. We want to treat the atom as system and single photon mode as environment, even though a single photon mode usually is not complex enough to be called "environment".

(2a) Our complete basis is $\{|\, g, n \,\rangle, |\, e, n \,\rangle$ for integer n. Show that in terms of this basis the Hamiltonian decomposes into 2 by 2 blocks and find those blocks explicitly in matrix form. If the initial state was $|\, e \,\rangle \otimes |\, 0 \,\rangle$, list all basis states that can become occupied in time. [2 points]
Solution: The first two operators are diagonal in terms of spin (electronic state) and oscillator state. The coupling operators act as $\hat{a}\hat{\sigma}_+ |\, g, n \,\rangle = \sqrt{n} |\, e, n-1 \,\rangle$*,* $\hat{a}\hat{\sigma}_+ |\, e, n \,\rangle = 0$*,*

[1] You do not yet need to know what these are, you will see them in PHY402.

$\hat{a}^\dagger\hat{\sigma}_-|\,g,n\,\rangle = 0$ *and* $\hat{a}^\dagger\hat{\sigma}_-|\,e,n\,\rangle = \sqrt{n+1}|\,g,n+1\,\rangle$. *We can interpret the first one as "excite the atom by absorption of one photon" and the second as "de-excite the atom through emission of one photon". Now,* $\hat{a}\hat{\sigma}_+|e,n-1\rangle = 0$, $\hat{a}^\dagger\hat{\sigma}_-|e,n-1\rangle = \sqrt{n}|g,n\rangle$. *So it appears that further operation of the coupling terms on the states that we got on the right hand side in the first applications, get us back to the original states.*

From the form of these coupling terms, we infer that the Hilbertspace is divided into subspaces with basis $B_n = \{|\,g,n+1\,\rangle, |\,e,n\,\rangle\}$, *for all* $n = 0,1,2,3\cdots$ *that do not couple to any other subspace* B_k $k \neq n$. *(Or you can explicitly find matrix elements between blocks and show that those are zero, as we had done in earlier assignments). This means that* $|\,g,0\,\rangle$ *does not couple to anything and is stationary.*

The Hamiltonian for block n is a 2 by 2 matrix (basis ordering $\{|\,g,n+1\,\rangle, |\,e,n\,\rangle\}$ *as above):*

$$\hat{H}_n = \begin{pmatrix} \hbar\omega(n+1) + \frac{\Delta E}{2} & \frac{\hbar\kappa}{2}\sqrt{n+1} \\ \frac{\hbar\kappa}{2}\sqrt{n+1} & \hbar\omega n - \frac{\Delta E}{2} \end{pmatrix} \tag{7}$$

If we start in $|\,e\,\rangle\otimes|\,0\,\rangle$ *(which is in the* $n = 0$ *block), the only relevant states are* $|\,e\,\rangle\otimes|\,0\,\rangle$ *and* $|\,g\,\rangle\otimes|\,1\,\rangle$ *(i.e. the atom is excited and there is no photon, and the atom is de-excited and there is one photon).*

(2b) Find the time evolution operator, and from that the evolution of the reduced density matrix for the atom from the initial state $|\,e\,\rangle\otimes|\,0\,\rangle$. [4 points]
Solution: Since the Hamiltonian decomposes into blocks, so does the time evolution operator (see earlier assignments). Each block is $\hat{U}_n = \exp[-i\hat{H}_n t/\hbar]$, *which we can find through matrix exponentiation:*

$$\hat{U}_n = \begin{pmatrix} \frac{e^{-i\omega(n+\frac{1}{2})t}\left[\cos(\omega_{\mathit{eff}}\,t/2)\omega_{\mathit{eff}} + i\sin(\omega_{\mathit{eff}}\,t/2)(\omega-\Delta E/\hbar)\right]}{\omega_{\mathit{eff}}} & -i\kappa\sqrt{n+1}\frac{e^{-i\omega(n+\frac{1}{2})t}\sin(\omega_{\mathit{eff}}\,t/2)}{\omega_{\mathit{eff}}} \\ -i\kappa\sqrt{n+1}\frac{e^{-i\omega(n+\frac{1}{2})t}\sin(\omega_{\mathit{eff}}\,t/2)}{\omega_{\mathit{eff}}} & \frac{e^{-i\omega(n+\frac{1}{2})t}\left[\cos(\omega_{\mathit{eff}}\,t/2)\omega_{\mathit{eff}} - i\sin(\omega_{\mathit{eff}}\,t/2)(\omega-\Delta E/\hbar)\right]}{\omega_{\mathit{eff}}} \end{pmatrix} \tag{8}$$

with $\omega_{\mathit{eff}} = \sqrt{(\omega - \Delta E/\hbar)^2 + (n+1)\kappa^2}$.

The initial state $|\,e\,\rangle\otimes|\,0\,\rangle$ *belongs to the block* $n = 0$, *so we only have to worry about a state of the form* $|\,\Psi(t)\,\rangle = c_g(t)|\,g,1\,\rangle + c_e(t)|\,e,0\,\rangle$, *with initial conditions* $\mathbf{c}(t) = [0,1]^T$. *Multiplying the matrix* $\hat{U}_0$ *with this column vector we find:*

$$\begin{aligned} |\,\Psi(t)\,\rangle = &-i\kappa\frac{e^{-i\frac{\omega}{2}t}\sin(\omega_{\mathit{eff}}\,t/2)}{\omega_{\mathit{eff}}}|\,g,1\,\rangle \\ &+ \frac{e^{-i\frac{\omega}{2}t}\left[\cos(\omega_{\mathit{eff}}\,t/2)\omega_{\mathit{eff}} - i\sin(\omega_{\mathit{eff}}\,t/2)(\omega-\Delta E/\hbar)\right]}{\omega_{\mathit{eff}}}|\,e,0\,\rangle \end{aligned} \tag{9}$$

The reduced density matrix from this sub-space will be

$$\hat{\rho}_{\mathcal{S}}(t) = |c_g(t)|^2|\,g\,\rangle\langle\,g\,| + |c_e(t)|^2|\,e\,\rangle\langle\,e\,|. \tag{10}$$

(2c) Now we want to use this example to understand Kraus operators. From (b) and the initial state $|\,e\,\rangle \otimes |\,0\,\rangle$, find the Kraus operators of the problem. Then use those to find the reduced density matrix of the atom and compare with your result from (b). [4 points]
Solution: The Kraus operators are defined through Eqn. (3.65) in the lecture:

$$\hat{E}_{ij}(t) = \sqrt{p_i}\langle\, E_j\,|\hat{U}(t)|\,E_i\,\rangle \tag{11}$$

where i, j label the environmental basis. It depends on the initial state of the environment through p_i. Here we have $p_i = \delta_{i0}$, i.e. only $i = 0$ will contribute, which helps a lot. In that case

$$\hat{E}_{0j}(t) = \langle\, E_j\,|\hat{U}(t)|\,E_0\,\rangle. \tag{12}$$

Due to the block structure of $\hat{U}$ the problem simplifies even further, we know $\hat{U}$ can only couple $|\,0\,\rangle$ of the environment to $|\,1\,\rangle$, hence only $j = 0$ and $j = 1$ are relevant and given by $\hat{E}_{00}(t) = \langle\, 0\,|\hat{U}_0(t)|\,0\,\rangle$, and $\hat{E}_{01}(t) = \langle\, 1\,|\hat{U}_0(t)|\,0\,\rangle$ in terms of $\hat{U}_n$ from Eq. (8).

Let us write the matrix in (8) as

$$\hat{U}_0 = U_{gg}|\,g,1\,\rangle\langle\,g,1\,| + U_{ge}|\,g,1\,\rangle\langle\,e,0\,| + U_{eg}|\,e,0\,\rangle\langle\,g,1\,| + U_{ee}|\,e,0\,\rangle\langle\,e,0\,|. \tag{13}$$

then $\hat{E}_{00}(t) = U_{ee}|\,e\,\rangle\langle\,e\,|$, and $\hat{E}_{01}(t) = U_{ge}|\,g\,\rangle\langle\,e\,|$.

Starting from $\hat{\rho}(0)_{\mathcal{S}} = |\,e\,\rangle\langle\,e\,|$, we can now use Eq. (3.66) of the lecture to find the reduced density matrix for the system at time t:

$$\begin{aligned}
\hat{\rho}(t)_{\mathcal{S}} &= \sum_{ij} \hat{E}_{ij}(t)\hat{\rho}(0)_{\mathcal{S}}\hat{E}_{ij}^{\dagger}(t) \\
&\overset{here}{=} U_{ee}|\,e\,\rangle\langle\,e\,|\Big(|\,e\,\rangle\langle\,e\,|\Big)U_{ee}^{*}|\,e\,\rangle\langle\,e\,| + U_{ge}|\,g\,\rangle\langle\,e\,|\Big(|\,e\,\rangle\langle\,e\,|\Big)U_{ge}^{*}|\,e\,\rangle\langle\,g\,| \\
&= |U_{ee}|^2|\,e\,\rangle\langle\,e\,| + |U_{ge}|^2|\,g\,\rangle\langle\,g\,| \\
&= \left[\cos^2(\omega_{\mathit{eff}}\,t/2) + \sin^2(\omega_{\mathit{eff}}\,t/2)\frac{(\omega - \Delta E/\hbar)^2}{\omega_{\mathit{eff}}^2}\right]|\,e\,\rangle\langle\,e\,| + (n+1)\left|\frac{\kappa\sin(\omega_{\mathit{eff}}\,t/2)}{\omega_{\mathit{eff}}}\right|^2|\,g\,\rangle\langle\,g\,|,
\end{aligned} \tag{14}$$

which is the same as what we had found in part (b).

(2d) (bonus) How do the Kraus operators change if we had a more complex initial state $|\,e\,\rangle \otimes \sum_n c_n|\,n\,\rangle$, with multiple non-zero entries. What does this imply for the system evolution?
Solution: We then get two copies of the Kraus operators found above from within each block of the time evolution operator (or each photon number state with initially nonzero occupation). Since the trigonometric functions in each have slightly different frequencies, oscillations coming from different blocks will dephase, so the end result might be less regularly periodic than that for the simple initial state above.

(3) Numerical solution of Jaynes-Cummings model: We now want to verify results from Q2 numerically, and also setup a tool that can solve the dynamics for more general cases.

(3a) Using the expression

$$|\,\psi(t)\,\rangle = \sum_{a\in\{g,e\},n} c_{an}(t)|\,e,n\,\rangle, \tag{15}$$

for the general time-dependent state expressed in the basis advocated in Q2, derive equations of motion for the coefficients $c_{an}(t)$ and implement those in the template `Assignment4_code_draft_v1.xmds`. Run this without changing parameters and as usual first check whether your equations conserve energy and normalisation of the state, using `Assignment4_plot_checks_v1.m`. [5 points]

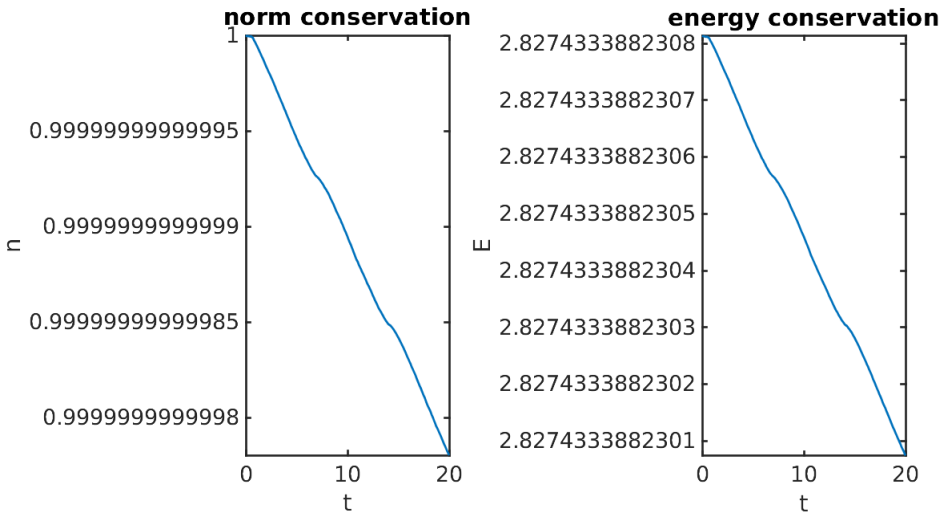
Solution: We insert Eq. (15) into $i\hbar|\,\dot{\psi}(t)\,\rangle = \hat{H}|\,\psi(t)\,\rangle$*, apply all ladder and spin operators onto basis vectors and then project onto* $\langle\, g,k\,|$ *and* $\langle\, e,k\,|$*, in the end we rename* $k \to n$*. Then we find:*

$$\begin{aligned}
\dot{c}_{en}(t) &= -i\left(+\frac{\Delta E}{2\hbar} + \omega n\right) c_{en} - i\frac{\kappa}{2}\sqrt{n+1}c_{g,n+1}, \\
\dot{c}_{gn}(t) &= -i\left(-\frac{\Delta E}{2\hbar} + \omega n\right) c_{gn} - i\frac{\kappa}{2}\sqrt{n}c_{e,n-1}.
\end{aligned} \tag{16}$$

We ignore terms with negative subscripts (these are multiplied with zero anyway). Eq. (16) conserves the norm $\langle\,\psi(t)\,|\,\psi(t)\,\rangle = \sum_n(|c_{gn}|^2 + |c_{en}|^2)$ *and energy*

$$\begin{aligned}
\langle\hat{H}\rangle = \sum_n \Big[&\frac{\Delta E}{2}(|c_{en}|^2 - |c_{gn}|^2) + \hbar\omega n(|c_{en}|^2 + |c_{gn}|^2) \\
&+ \hbar\kappa\sqrt{n+1}\ Re(c^*_{en}c_{g,n+1}) + \hbar\kappa\sqrt{n}\ Re(c^*_{gn}c_{e,n-1})\Big]
\end{aligned} \tag{17}$$

up to numerical precision as shown in the figure below.



(3b) The code is set up to sample the reduced density matrix of the atom in two different way, once directly from the time evolving state (you can rederive this for practice if you like) and once via the Kraus operators. You still have to insert your result for the Kraus operator from Q2 into the code at the indicated position. Finally use the script `Assignment4_compare_reddm_v1.m` to compare both approaches (verifying your Kraus operators from Q2). Discuss why the reduced DM evolves in the way it does. [5 points]
Solution: The complete comparison of the two density matrices is shown below, (K) mark lines where it has been obtained via the Kraus operators. We used parameters ($\hbar = 1$) with $\Delta E = 0.9(2\pi)$, $\kappa = 0.1(2\pi)$, $\omega = (2\pi)$. In that case ω_{eff} for the block $n = 0$ in Eq. (10) is equal to $\omega_{eff} = \sqrt{[0.1(2\pi)]^2 + [0.1(2\pi)]^2} = \sqrt{2}\kappa$ and the maximal population ever to get into the state $|\,g,1\,\rangle$ is $|\kappa/\omega_{eff}|^2 = 1/2$. We just see detuned periodic Rabi oscillations between the initial state $|\,e,0\,\rangle$ and $(|\,e,0\,\rangle + |\,g,1\,\rangle)/\sqrt{2}$. When tracing over the environment the latter gives us a 50-50 mixture for the atom's electronic state, which is what is shown in the figure.