Phys 637, I-Semester 2022/23, Assignment 3

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Due-date: Lecture 15.9.2021

(1) Pointer states:

(1a) What are the pointer states of the system in the quantum brownian motion Hamiltonian? [2pts].

(1b) Invent your own (ideally quite simple) Hamiltonian for (i) a system, (ii) a measurement apparatus capable of measuring a certain observable of the system, (iii) and environment, for simplicity only interacting with the apparatus. In terms of that Hamiltonian, decide what are the pointer states of the apparatus [4pts].

(2) Von-Neumann measurement of position: Consider a particle in one dimension (1D) with wavefunction $\psi(x_1) = \frac{1}{\pi^{1/4}\sigma^{1/2}} e^{-\frac{x_1^2}{2\sigma^2}}$ and a second "detector or apparatus" particle (also in 1D) with some unknown initial wavefunction $\phi_0(x_2)$. Suppose through some sort of interaction, the initial to final state sequence of this system plus apparatus model is

$$\Psi_0(x_1, x_2) = \psi(x_1)\phi_0(x_2) \rightarrow \Psi_f(x_1, x_2) = \psi(x_1)\phi(x_2 - x_1), \tag{1}$$

where $\phi(x_2) = \frac{1}{\pi^{1/4}\sigma^{1/2}} e^{-\frac{x_2^2}{2\eta^2}}$, with $\eta \ll \sigma$. Now suppose we measure the position x_2 of the second particle with finite resolution r_2. Our measurement apparatus has a series of pixels i, and one can say that after finding a click on pixel j the wavefunction has collapsed into $\hat{P}_j | \Psi \rangle$, where

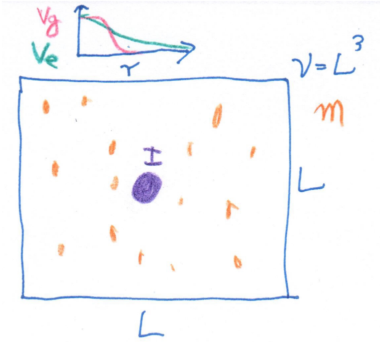
$$\hat{P}_j = \int dx' f_j(x') | x' \rangle \langle x' |, \tag{2}$$

where $f_j(x') > 0$ are real $\sum_j f_j(x') = 1$ (*Think of each f_j as the sensitivity for (discrete) pixel x_j to trigger if the (continuous) position was near x_j. For a perfect detector this should be a step function, but that does not exist, so at best it will be a smoothened/blurry step function).* The $\hat{P}_j$ are called a "positive operator-valued measure" (POVM), and can be applied to describe realistic physical measurements of continuum variables, which cannot be perfect projections onto just one state.

(2a) Make a drawing of the density $|\Psi_f(x_1, x_2)|^2$ prior to measurement, discussing its physical meaning. Then also draw the density after our detector has measured position on pixel j. Use this to argue why we have made an imperfect QND measurement of the position of particle 1. [4 points] *Hint: To convert a wavefunction $\varphi(x)$ of position into bra-ket notation, use $| \Psi \rangle = \int dx \varphi(x) | x \rangle$ and then $\langle x_1 | x_2 \rangle = \delta(x_1 - x_2)$ for position eigenkets, see the new section 3.1.3. of the notes.*

(2b) Now bring the initial two-body wavefunction into momentum space with a Fourier transform, and draw or plot $|\tilde{\Psi}_f(p_1, p_2)|^2$ (taking into account $\eta \ll \sigma$). Discuss whether the detector particle could also be viewed as having measured the momentum p_x of the other particle while you treat it as a von-Neumann measurement. Relate your discussion to the preferred basis problem. [5 points]

(3) (Decoherence) Consider an infinitely heavy two-level impurity atom (violet) at the origin, within a very cold gas of N free atoms of mass m from another species, with which it interacts through central interaction potentials $V_g(r)$ ($V_e(r)$) if it is in state $|\, g\,\rangle$ ($|\, e\,\rangle$), as sketched below:



All the (mutually non-interacting) ambient atoms (orange) are in the zero-momentum ground state of a box shaped quantisation volume $\mathcal{V}$, hence their wavefunction is $\phi(\mathbf{x}) = \langle\, \mathbf{x}\,|\, 0\,\rangle = 1/\sqrt{\mathcal{V}} = const.$

We can write the Hamiltonian for this system as:

$$\hat{H} = \Delta E\big[|\, e\,\rangle\langle\, e\,|\big]^{(I)} + \sum_n \Bigg[\left(-\frac{\hbar^2 \boldsymbol{\nabla}_{\mathbf{r}_{(n)}}{}^2}{2m} \right) + V_e(|\mathbf{r}_{(n)}|)\big[|\, e\,\rangle\langle\, e\,|\big]^{(I)} + V_g(|\mathbf{r}_{(n)}|)\big[|\, g\,\rangle\langle\, g\,|\big]^{(I)} \Bigg], \tag{3}$$

where $\mathbf{r}_{(n)}$ is the position of gas atom n and operators with $^{(I)}$ act on the impurity only, and the energy difference between $|\, g\,\rangle$ and $|\, e\,\rangle$ is ΔE.

(3a) Show or argue that this Hamiltonian decomposes into lots of blocks, one for each environment atom and impurity spin state [2pts].

(3b) We assume the initial system-environment state

$$|\, \Psi\,\rangle = \frac{1}{\sqrt{2}}(|\, g\,\rangle + |\, e\,\rangle) \otimes \prod_k [|\, 0\,\rangle]^{(k)}, \tag{4}$$

where $^{(k)}$ flags the state of environment atom k. Show, that within each block, we only have to solve a single particle problem, described by the Schrödinger equation:

$$i\hbar\dot{\phi}_k(\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_k(\mathbf{x}) + \Delta E \delta_{ke} \right) \phi_k(\mathbf{x}), \tag{5}$$

where k labels the impurity state. For short times and very strong potential V (such that we can ignore the kinetic energy operator), show that this is solved approximately by $\phi_k(\mathbf{x}, t) = \phi_k(\mathbf{x}, 0)e^{-\frac{i}{\hbar}[V_k(\mathbf{x})+\Delta E\delta_{ke}]t}$ [2pts].

(3c) While these conditions are valid (for short times), find the time-evolution of the magnitude of the coherence between $|\, g\,\rangle$ and $|\, e\,\rangle$ in the reduced density matrix of the system [5pts].

(4) Numerical evaluation of Wigner function:

(4a) Show the two properties of the Wigner function, that $P(x) = \int dpW(x,p)$ and $P(p) = \int dxW(x,p)$, where $P(x)$ [$P(p)$] is the position (momentum) distribution in state $\rho = |\, \Psi\,\rangle\langle\, \Psi\,|$. [3 points]

(4b) The script `Assignment3_plot_wignerfunction.m` can be used to plot the Wigner function of any input state $\Psi(x)$ in matlab. Compare the wigner function in (i) a single coherent state, (ii) a specific oscillator eigenstate, e.g. $n = 5$, (iii) a superposition of two oscillator eigenstates. Discuss in the context of your understanding of a classical phase space. Then edit the script to plot Wigner functions of any other 1D quantum states that interest you (discuss those as well). [3 points]