

Phys 637, I-Semester 2022/23, Assignment 3 solution

(1) Pointer states:

(1a) What are the pointer states of the system in the quantum brownian motion Hamiltonian? [2pts].

Solution: The interaction Hamiltonian of the QBM model is:

$$H_{int} = \hat{X} \otimes \sum_i \kappa_i q_i = \hat{X} \otimes \hat{E} \quad (1)$$

So the pointer states of the system are the eigenfunctions of the \hat{X} operator which is the set of delta functions $\{\delta(x - a), \forall a \in \mathcal{R}\}$.

(1b) Invent your own (ideally quite simple) Hamiltonian for (i) a system, (ii) a measurement apparatus capable of measuring a certain observable of the system, (iii) and environment, for simplicity only interacting with the apparatus. In terms of that Hamiltonian, decide what are the pointer states of the apparatus [4pts].

Solution: Rydberg atom is an atom with a very high principal quantum number meaning that the valence electrons (usually a single electron) will have a large orbital size. Consider a Rydberg atom embedded with a fixed position within a Bose-Einstein condensate. In such a case, one can consider the Rydberg atom as a system of a pseudospin (where the angular momentum state $|s(l = 0)\rangle \rightarrow |\downarrow\rangle$ and $|p(l = 1)\rangle \rightarrow |\uparrow\rangle$), the BEC atoms inside the Rydberg orbit (of radius r) form the apparatus and the BEC atoms outside the Rydberg orbit form the environment. In this case, the Rydberg electron interacts only with the atoms within its orbital range as such the environment interacts only with the apparatus. For simplicity, we consider a single bosonic oscillator of frequency ω_A within the Rydberg atom orbit as the apparatus (coupled to the system via the coupling constant κ which depends on l such that the atom can “measure” the pseudospin state) and another single oscillator of frequency ω_E outside it as the environment (interaction strength between the apparatus and environment is U_0). Then the different parts of the Hamiltonian can be written as follows:

$$\hat{H}_S = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z - \frac{1}{2}\hbar\Delta\hat{\sigma}_x \quad (2)$$

$$\hat{H}_A = \frac{p_A^2}{2m_A} + \frac{1}{2}m_A\omega_A^2 r_A^2 \quad (3)$$

$$\hat{H}_{SA} = \hat{\sigma}_z \otimes \kappa \hat{r}_A \quad (4)$$

$$\hat{H}_E = \frac{p_E^2}{2m_E} + \frac{1}{2}m_E\omega_E^2 (r_E - r)^2 \quad (5)$$

$$\hat{H}_{AE} = U_0 \hat{r}_A \otimes \hat{r}_E \quad (6)$$

Then the pointer states of the apparatus will be the position eigenstates of an oscillator.

(2) Von-Neumann measurement of position: Consider a particle in one dimension

(1D) with wavefunction $\psi(x_1) = \frac{1}{\pi^{1/4}\sigma^{1/2}}e^{-\frac{x_1^2}{2\sigma^2}}$ and a second “detector or apparatus” particle (also in 1D) with some unknown initial wavefunction $\phi_0(x_2)$. Suppose through some sort of interaction, the initial to final state sequence of this system plus apparatus model is

$$\Psi_0(x_1, x_2) = \psi(x_1)\phi_0(x_2) \rightarrow \Psi_f(x_1, x_2) = \psi(x_1)\phi(x_2 - x_1), \quad (7)$$

where $\phi(x_2) = \frac{1}{\pi^{1/4}\sigma^{1/2}}e^{-\frac{x_2^2}{2\eta^2}}$, with $\eta \ll \sigma$. Now suppose we measure the position x_2 of the second particle with finite resolution r_2 . Our measurement apparatus has a series of pixels i , and one can say that after finding a click on pixel j the wavefunction has collapsed into $\hat{P}_j|\Psi\rangle$, where

$$\hat{P}_j = \int dx' f_j(x')|x'\rangle\langle x'|, \quad (8)$$

where $f_j(x') > 0$ are real $\sum_j f_j(x') = 1$ (*Think of each f_j as the sensitivity for (discrete) pixel j to trigger if the (continuous) position was x' . For a perfect detector this should be a step function, but that does not exist, so at best it will be a smoothed/blurry step function*). The \hat{P}_j are called a “positive operator-valued measure” (POVM), and can be applied to describe realistic physical measurements of continuum variables, which cannot be perfect projections onto just one state.

(2a) Make a drawing of the density before the measurement $|\Psi_f(x_1, x_2)|^2$, discussing its physical meaning. Then also draw the density after our detector has measured position on pixel j . Use this to argue why we have made an imperfect QND measurement of the position of particle 1. [4 points] *Hint: To convert a wavefunction $\varphi(x)$ of position into bra-ket notation, use $|\Psi\rangle = \int dx\varphi(x)|x\rangle$ and then $\langle x_1|x_2\rangle = \delta(x_1 - x_2)$ for position eigenkets.*

Solution:

$$\begin{aligned} |\Psi_f(x_1, x_2)|^2 &= |\Psi(x_1)\phi(x_2 - x_1)|^2 \\ &= N \exp\left[-\frac{x_1^2}{\sigma^2} - \frac{(x_2 - x_1)^2}{\eta^2}\right] ; \eta \ll \sigma, \\ &= N \exp\left[-\frac{x_1^2}{\sigma^2}\right] \exp\left[-\frac{(x_2 - x_1)^2}{\eta^2}\right]. \end{aligned} \quad (9)$$

This means particle 1 (x_1) is in a Gaussian state of width σ and particle 2 (x_2) is “close to x_1 ” up to a distance η . The final state density $|\Psi_f(x_1, x_2)|^2$ is shown in Fig. 1. The joint probability density to find particle 1 in an interval dx_1 around position x_1 and particle 2 in an interval dx_2 around position x_2 is given by $|\Psi_f(x_1, x_2)|^2 dx_1 dx_2$. Hence we can see from the diagram that measuring particle one near some $x_1 = x_0$, will very likely imply that particle 2 is also at $x_2 = x_0$, up to a deviation of order η . Hence we can view the second particle as a “von-Neumann-measurement apparatus” that has measured the position of

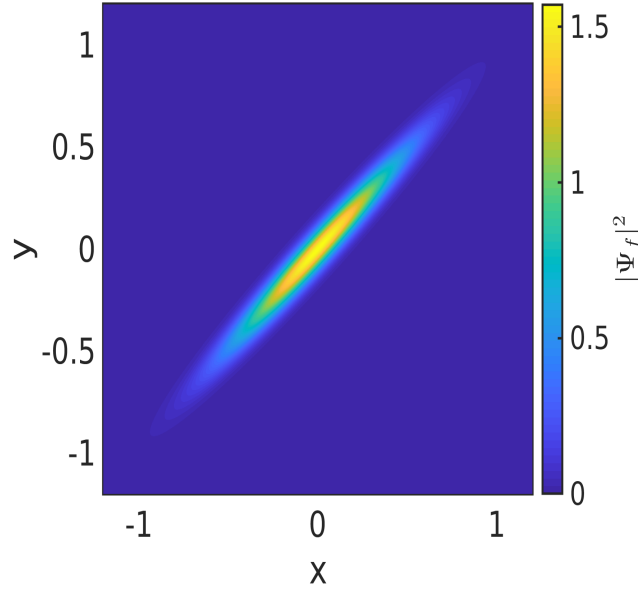


Figure 1: Position space density $|\Psi_f(x, y)|^2$ after interactions before projection.

particle 1 (or the other way round). To apply the projection operator, we use the hint for the 2-dimensional case as follows:

$$|\Psi_f\rangle = \int dx_1 dx_2 \Psi_f(x_1, x_2) |x_1 x_2\rangle. \quad (10)$$

Hence,

$$\begin{aligned} \hat{P}_j |\Psi\rangle &= \int dx_1 dx_2 \Psi_f(x_1, x_2) \int dx' f_j(x') |x_1\rangle |x'\rangle \underbrace{\langle x' | x_2 \rangle}_{\delta(x' - x_2)} \\ &= \int dx_1 dx_2 \Psi_f(x_1, x_2) f_j(x_2) |x_1\rangle |x_2\rangle. \end{aligned} \quad (11)$$

So, the post-measurement wavefunction can be written as:

$$\Psi_{pm}(x_1, x_2) = \Psi_f(x_1, x_2) f_j(x_2). \quad (12)$$

The density plot of $|\Psi_{pm}(x_1, x_2)|^2$ is as given in Fig.2.

(2b) Now bring the initial two-body wavefunction into momentum space with a Fourier transform, and draw or plot $|\tilde{\Psi}_f(p_1, p_2)|^2$ (taking into account $\eta \ll \sigma$). Discuss whether the detector particle could also be viewed as having measured the momentum p_1 of the other particle while you treat it as a van-Neumann measurement. Relate your discussion to the preferred basis problem. [5 points]

Solution: The Fourier transformation of the final wavefunction can be calculated with the

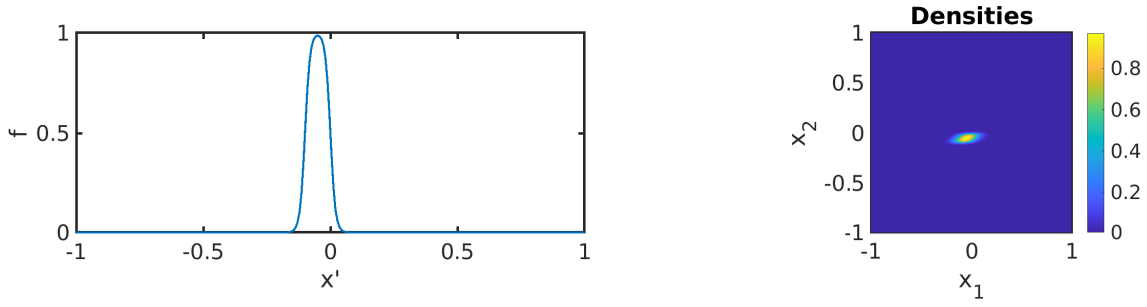


Figure 2: Position space density $|\Psi_f(x, y)|^2$ after projection (complete measurement).

usual methods as:

$$\begin{aligned}
\tilde{\Psi}_f(p_1, p_2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \Psi(x_1, x_2) e^{-ip_1 x_1/\hbar - ip_2 x_2/\hbar}, \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \psi(x_1) \phi(x_2 - x_1) e^{-ip_1 x_1/\hbar - ip_2 x_2/\hbar}, \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\frac{x_1^2}{2\sigma^2}} e^{-\frac{(x_2 - x_1)^2}{2\eta^2}} e^{-ip_1 x_1/\hbar - ip_2 x_2/\hbar}. \quad (13)
\end{aligned}$$

This integral can be calculated either using a Gaussian integration technique or with mathematica. Using the Mathematica, we get:

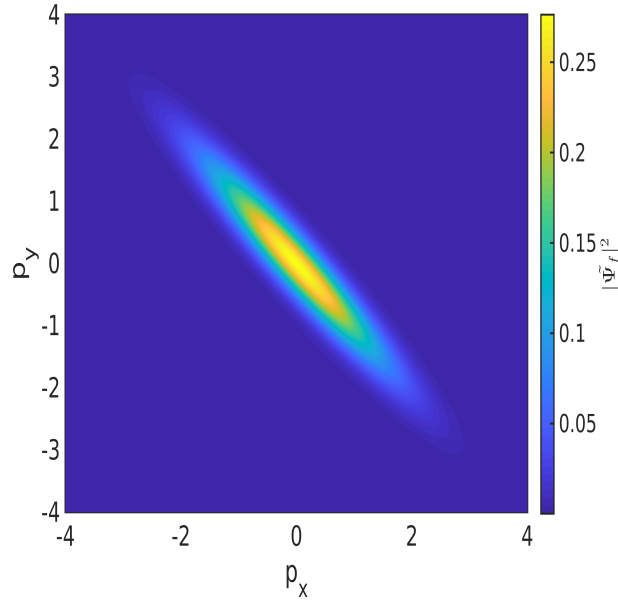


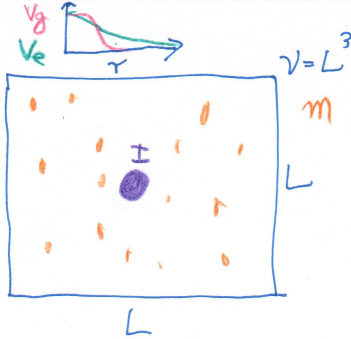
Figure 3: Momentum space density $|\tilde{\Psi}_f(p_x, p_y)|^2$ after interactions before projection.

$$\tilde{\Psi}_f(p_1, p_2) = \frac{1}{\sqrt{\pi}\sigma} \frac{1}{\sqrt{\frac{1}{\eta^2} + \frac{1}{\sigma^2}}} \frac{1}{\sqrt{\frac{1}{\eta^2 + \sigma^2}}} \times e^{\left(\frac{-p_2^2 \sigma^2 - 2p_1 p_2 \sigma^2 - p_1^2 (\eta^2 + \sigma^2)}{2\hbar^2}\right)},$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\pi}\sigma} \frac{1}{\sqrt{\frac{1}{\eta^2} + \frac{1}{\sigma^2}}} \frac{1}{\sqrt{\frac{1}{\eta^2 + \sigma^2}}} \times e^{\left(\frac{-(p_1+p_2)^2\sigma^2 - p_2^2\eta^2}{2\hbar^2}\right)}, \\
&= \frac{\sqrt{\eta^2}}{\sqrt{\pi}} \times e^{\left(\frac{-(p_1+p_2)^2\sigma^2 - p_2^2\eta^2}{2\hbar^2}\right)}, \\
&= \tilde{\psi}(p_1 + p_2)\tilde{\phi}(p_2), \tag{14}
\end{aligned}$$

where $\tilde{\psi}(p_1 + p_2) = \frac{\sqrt{\eta}}{\pi^{1/4}} e^{\left(\frac{-(p_1+p_2)^2\sigma^2}{2\hbar^2}\right)}$ and $\tilde{\phi}(p_2) = \frac{\sqrt{\eta}}{\pi^{1/4}} e^{\left(\frac{-(p_2)^2\eta^2}{2\hbar^2}\right)}$. The final density, $|\tilde{\Psi}(p_1, p_2)|^2$ is shown in Fig. 3 for some chosen parameters under the condition $\eta \ll \sigma$. We can see that if we measure particle 2 to have some momentum $p_2 = p_0$, then very likely particle one had momentum $p_1 = -p_0$. Except the sign flip the situation is thus the same as for position, and we can also interpret particle 2 as having measured the momentum of particle 1. Treated as a von-Neumann-measurement, we can thus see that we cannot even tell whether our ‘‘apparatus’’ measures position or momentum, even though the two are conjugate variables. This is a manifestation of the preferred basis problem discussed in the lecture, for a continuous degree of freedom.

(3) (Decoherence) Consider an infinitely heavy two-level impurity atom (violet) at the origin, within a very cold gas of N free atoms of mass m from another species, with which it interacts through central interaction potentials $V_g(r)$ ($V_e(r)$) if it is in state $|g\rangle$ ($|e\rangle$), as sketched below:



All the (mutually non-interacting) ambient atoms (orange) are in the zero-momentum ground state of a box shaped quantisation volume \mathcal{V} , hence their wavefunction is $\phi(\mathbf{x}) = \langle \mathbf{x} | 0 \rangle = 1/\mathcal{V} = \text{const}$.

We can write the Hamiltonian for this system as:

$$\begin{aligned}
\hat{H} = \Delta E [|e\rangle \langle e|]^{(I)} + \sum_n \left[\left(-\frac{\hbar^2 \nabla_{\mathbf{r}(n)}^2}{2m} \right) \right. \\
\left. + V_e(|\mathbf{r}(n)|) [|e\rangle \langle e|]^{(I)} + V_g(|\mathbf{r}(n)|) [|g\rangle \langle g|]^{(I)} \right], \tag{15}
\end{aligned}$$

where $\mathbf{r}(n)$ is the position of gas atom n and operators with (I) act on the impurity only, and the energy difference between $|g\rangle$ and $|e\rangle$ is ΔE .

(3a) Show or argue that this Hamiltonian decomposes into lots of blocks, one for each environment atom and impurity spin state [2pts].

(3b) We assume the initial system-environment state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes \prod_k [|0\rangle]^{(k)}, \quad (16)$$

where $^{(k)}$ flags the state of environment atom k . Show, that within each block, we only have to solve a single particle problem, described by the Schrödinger equation:

$$i\hbar\dot{\phi}(\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_k(\mathbf{x}) \right) \phi(\mathbf{x}), \quad (17)$$

where k labels the impurity state. For short times and very strong potential V (such that we can ignore the kinetic energy operator), show that this is solved approximately by $\phi(\mathbf{x}, t) = \phi(\mathbf{x}, 0)e^{-\frac{i}{\hbar}V_k(\mathbf{x})t}$ [2pts]. *Solution: Due to the block structure above, the time evolution operator decomposes also into*

$$\hat{U}(t) = \hat{U}_g(t)|g\rangle\langle g| + \hat{U}_e(t)|e\rangle\langle e|. \quad (18)$$

and $\hat{U}_g(t) = \prod_k \hat{U}_g^{(k)}(t)$, with $\hat{U}_g^{(k)}(t) = e^{-i\langle g|\hat{h}^{(n)}|g\rangle/ht}$. That last part is just a single particle time evolution operator, containing a potential energy given by $V_g(\mathbf{x})$. This, we know from QM1, can be handled by the TDSE Eq. (17). If we ignore the kinetic energy operator, that equation can just be trivially solved for each \mathbf{r} separately, with the solution given. (Or insert that solution into TDSE, to show that it solves it).

(3c) While these conditions are valid (for short times), find the time-evolution of the magnitude of the coherence between $|g\rangle$ and $|e\rangle$ in the reduced density matrix of the system [5pts]. *Solution: Applying this to the initial state Eq. (19) gives us the time evolving state:*

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|g\rangle \otimes \prod_k [|\phi_g(t)\rangle]^{(k)} + |e\rangle \otimes \prod_k [|\phi_e(t)\rangle]^{(k)}), \quad (19)$$

where $\phi_k(\mathbf{x}, t) = \langle \mathbf{x} | \phi_k(t) \rangle$.

Using Eq. (3.21) or (3.28) of the lecture, we now know that the coherence between $|g\rangle$ and $|e\rangle$ in the reduced density matrix of the system is

$$\begin{aligned} \rho_{eg} &= \frac{1}{2} \left(\prod_k \langle \phi_g(t) | \right)^{(k)} \left(\prod_k | \phi_e(t) \rangle \right)^{(k)} = \frac{1}{2} \prod_k \langle \phi_g(t) | \phi_e(t) \rangle^{(k)} \\ &= \left(\int_{\mathcal{V}} d^3\mathbf{x} \phi_g(\mathbf{x}, t)^* \phi_e(\mathbf{x}, t) \right)^N, \end{aligned} \quad (20)$$

where we have used the fact that the wavefunction for all N gas particles will have exactly the same time evolution, conditional only on the impurity being in g or e .

Further

$$\begin{aligned}
\rho_{eg} &= \left(\int_{\mathcal{V}} d^3\mathbf{x} \phi^*(\mathbf{x}, 0) e^{\frac{i}{\hbar} V_g(\mathbf{x})t} \phi(\mathbf{x}, 0) e^{-\frac{i}{\hbar} V_e(\mathbf{x})t} \right)^N, \\
&\approx \left(\frac{1}{\mathcal{V}} \int_{\mathcal{V}} d^3\mathbf{x} \left(1 + \frac{i}{\hbar} V_g(\mathbf{x})t \right) \left(1 - \frac{i}{\hbar} V_e(\mathbf{x})t \right) \right)^N \approx \left(1 + \frac{i}{\hbar\mathcal{V}} \int_{\mathcal{V}} d^3\mathbf{x} [V_g(\mathbf{x})t - V_e(\mathbf{x})t] \right)^N \\
&\approx 1 + \frac{iNt}{\hbar\mathcal{V}} \int_{\mathcal{V}} d^3\mathbf{x} [V_g(\mathbf{x}) - V_e(\mathbf{x})] \tag{21}
\end{aligned}$$

Using the gas particle density $\rho = N/\mathcal{V}$ and $|\rho_{eg}| = \sqrt{\rho_{eg}^* \rho_{eg}}$, we finally have:

$$|\rho_{eg}| \approx \frac{1}{2} \sqrt{1 - (\rho\Delta V/\hbar)^2 t^2} \approx \frac{1}{2} \left(1 - (\rho\Delta V/\sqrt{2}\hbar)^2 t^2 \right) = \frac{1}{2} (1 - (t/\tau)^2) \tag{22}$$

with $\Delta V \equiv \int_{\mathcal{V}} d^3\mathbf{x} [V_g(\mathbf{x}) - V_e(\mathbf{x})]$, from which we can read off the decoherence time-scale

$$\tau = \sqrt{2}\hbar/(\rho\Delta V), \tag{23}$$

similar to example 22 of the lecture. It seems logical, that decoherence is faster, if the gas is denser, and if the interaction with different states of the impurity are more different.

(4) Numerical evaluation of Wigner function:

(4a) Show the two properties of the Wigner function, that $P(x) = \int dp W(x, p)$ and $P(p) = \int dx W(x, p)$, where $P(x)$ [$P(p)$] is the position (momentum) distribution in state $\rho = |\Psi\rangle\langle\Psi|$. [3 points]

Solution: Starting from the Wigner function for a pure state

$$W(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{ipy} \Psi(x - \frac{y}{2}) \Psi(x + \frac{y}{2})^*, \tag{24}$$

we write

$$\begin{aligned}
\int_{-\infty}^{\infty} dp W(x, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \underbrace{\left[\int_{-\infty}^{\infty} dp e^{ipy} \right]}_{=(2\pi)\delta(y)} \Psi(x - \frac{y}{2}) \Psi(x + \frac{y}{2})^* = |\Psi(x)|^2 = P(x).
\end{aligned} \tag{25}$$

The other relation is slightly more tricky:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx W(x, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx e^{ipy} \Psi(x - \frac{y}{2}) \Psi(x + \frac{y}{2})^* \\
&\stackrel{\tilde{x}=x+y/2}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\tilde{x} e^{ipy} \Psi(\tilde{x} - y) \Psi(\tilde{x})^* \\
&\stackrel{\tilde{y}=\tilde{x}-y}{=} (-1)^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tilde{y} \int_{-\infty}^{\infty} d\tilde{x} e^{ip(\tilde{x}-\tilde{y})} \Psi(\tilde{y}) \Psi(\tilde{x})^* \\
&= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tilde{y} e^{-ip\tilde{y}} \Psi(\tilde{y}) \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tilde{x} e^{-ip\tilde{x}} \Psi(\tilde{x}) \right]^* = |\tilde{\Psi}(p)|^2 = P(p), \tag{26}
\end{aligned}$$

where we used the Fourier transform $\tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ipx} \Psi(x)$ and set $\hbar = 1$.

(4b) The script `Assignment3_plot_wignerfunction.m` can be used to plot the Wigner function of any input state $\Psi(x)$ in matlab. Compare the wigner function in (i) a single coherent state, (ii) a specific oscillator eigenstate, e.g. $n = 5$, (iii) a superposition of two oscillator eigenstates. Discuss in the context of your understanding of a classical phase space. Then edit the script to plot Wigner functions of any other 1D quantum states that interest you (discuss those as well). [3 points] *Solution: Figures for states (ii) and (iii) are shown below, state 1 is shown in example 25 of the lecture.*

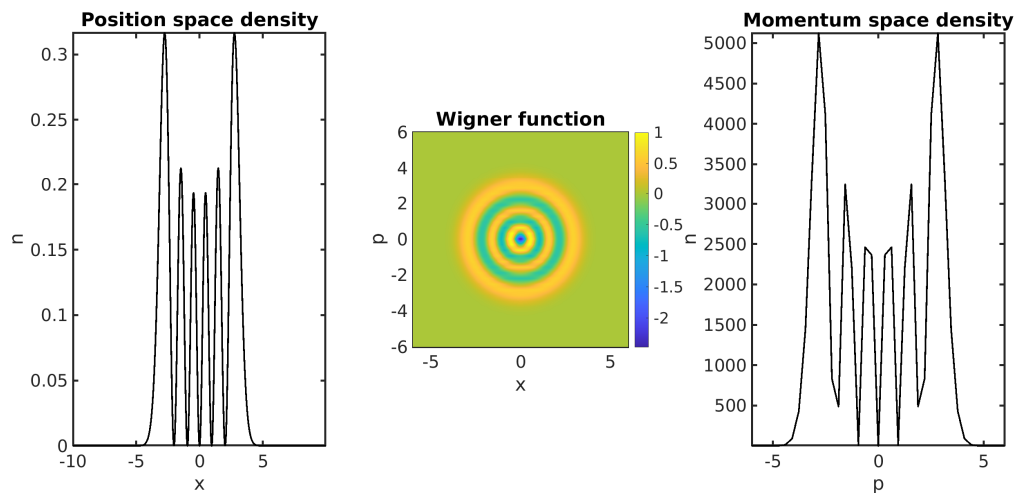


Figure 4: Position distribution (left), Wigner function (middle), and the momentum space distribution (right) for state 3 (harmonic oscillator state $\varphi_5(x)$) defined in the matlab script provided. We see an overall oscillatory structure. It is clear that negative regions in $W(x, p)$ are required, in order for the integrations as in (3a) to correctly reproduce the nodes with $P(x) = 0$ or $P(p) = 0$ in the position and momentum distribution.

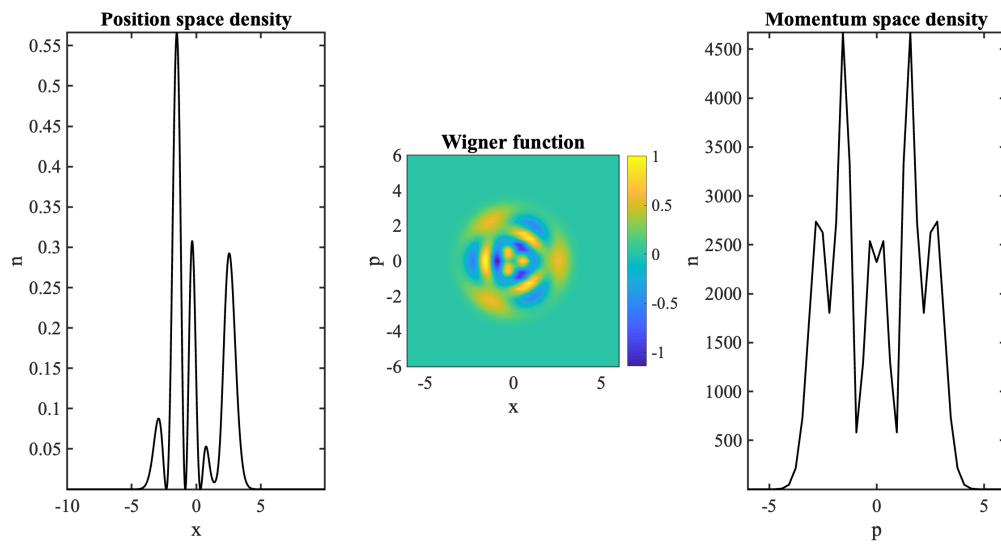


Figure 5: Position distribution (left), Wigner function (middle), and the momentum space distribution (right) for state 2, (superposition of harmonic oscillator state $\varphi_5(x)$ and $\varphi_2(x)$). We see an overall oscillatory structure with greater number of nodes, some rings at the right energy for either state, but many interference features.