Week 8

PHY 435 / 635 Decoherence and Open Quantum Systems
Instructor: Sebastian Wüster, IISER Bhopal, 2018

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4.5 Spin Decoherence

In section 4.4 we have applied the concepts of section 4 to the oscillator-oscillators model of section 2. We now do the same for the spin-oscillators model / Spin-Boson model.

4.5.1 Master equation for Spin-Boson model

Compared to our earlier treatment in section 2.2.1 we will at first not making the simplification $\Delta_0 = 0$, but instead set $\omega_0 = 0$. The complete Hamiltonian is then ($\hbar = 1$):

$$\hat{H} = \underbrace{-\frac{1}{2}\Delta_0\hat{\sigma}_x + \sum_i \omega_i \left(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right)}_{=\hat{H}_0} + \hat{\sigma}_z \otimes \sum_i \bar{\kappa}_i \left(\hat{a}_i + \hat{a}_i^\dagger\right), \tag{4.58}$$

Since the environment of the Spin-Boson model is the same as that for quantum Brownian motion, our results from section 4.4 regarding environmental correlations functions and noise Kernels, e.g. Eq. (4.32) and (4.36), can all be used here too.

What changes is mainly the system evolution, which we require in order to assemble the $\hat{S}^{(I)}(\tau)$ piece of the Master equation. We have $\hat{S} = \hat{\sigma}_z$ and hence

$$\begin{aligned}\hat{S}^{(I)}(\tau) = \hat{\sigma}_z(\tau) = e^{i\hat{H}_S\tau}\hat{\sigma}_z e^{-i\hat{H}_S\tau} &= e^{-i\Delta_0\hat{\sigma}_x\tau/2}\hat{\sigma}_z e^{i\Delta_0\hat{\sigma}_x\tau/2} \\ &\overset{Eq.\ (4.60)}{=} \hat{\sigma}_z(0)\cos\left(\Delta_0\tau\right) - \hat{\sigma}_y(0)\sin\left(\Delta_0\tau\right). \end{aligned} \tag{4.59}$$

Expression such as (4.59) occur frequently in the quantum dynamics involving spins and can be dealt with using the

Spin rotation formula: (related to Rodrigues' rotation formula)

$$e^{ia(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})}\boldsymbol{\sigma}e^{-ia(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})} = \boldsymbol{\sigma}\cos(2a) + (\hat{\mathbf{n}}\times\boldsymbol{\sigma})\sin(2a) + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})(1-\cos(2a)), \tag{4.60}$$

where $\hat{\mathbf{n}}$ is a unit vector that can be thought of as a rotation axis, and $2a$ a number that can be thought of as a rotation angle around that axis. $\boldsymbol{\sigma} = [\underline{\underline{\sigma}}_x, \underline{\underline{\sigma}}_y, \underline{\underline{\sigma}}_z]^T$ is a vector of Pauli matrices. The formula then gives the effect of that rotation on the cartesian spin components.

- To apply (4.60) to (4.59) (exercise) we only look at the z component of the transformed Pauli-vector, and set $a = -\Delta_0\tau/2$ and $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ (unit vector along x axis).

We can also recycle (4.37) from section 4.4 and just replace $\hat{S}$ with $\hat{\sigma}_z$. Then we insert (4.59) do some extensive re-arrangements, and reach the

Born-Markov master equation for the Spin-Boson model (with $\omega_0 = 0$):

$$\frac{d}{dt}\hat{\rho}_{\mathcal{S}}(t) = -i\big(\hat{H}'_{\mathcal{S}}\hat{\rho}_{\mathcal{S}}(t) - \hat{\rho}_{\mathcal{S}}(t)\hat{H}'^{\dagger}_{\mathcal{S}}\big) - \tilde{D}\big[\hat{\sigma}_z,\big[\hat{\sigma}_z,\hat{\rho}_{\mathcal{S}}(t)\big]\big]$$
$$+\zeta\hat{\sigma}_z\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_y + \zeta^*\hat{\sigma}_y\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_z, \tag{4.61}$$

where all operators without time argument are at $t = 0$. The Lamb-shifted Hamiltonian is

$$\hat{H}'_{\mathcal{S}} = \left(-\frac{1}{2}\Delta_0 - \zeta^*\right)\hat{\sigma}_x, \tag{4.62}$$

and the coefficients are

$$\zeta = \tilde{f} - i\tilde{\gamma}, \tag{4.63}$$
$$\tilde{D} = \int_0^\infty d\tau\ \nu(\tau)\cos(\Delta_0\tau), \tag{4.64}$$
$$\tilde{f} = \int_0^\infty d\tau\ \nu(\tau)\sin(\Delta_0\tau), \tag{4.65}$$
$$\tilde{\gamma} = \int_0^\infty d\tau\ \eta(\tau)\sin(\Delta_0\tau). \tag{4.66}$$

- We use $\tilde{..}$ on the coefficients to make apparent that they are not quite the same as we had for quantum Brownian motion.
- Since $\zeta \in \mathbb{C}$ in general, $\hat{H}'_{\mathcal{S}}$ may no longer be Hermitian.
- The $\tilde{D}$ term already explicitly takes the Lindblad form as in Eq. (4.25). This is the most important term. To see its function, let's look at it in detail:

4.5.2 Decoherence in the Spin-Boson model

Luckily, with the simplification $\Delta_0 = 0$ all coefficients in (4.61) drastically simplify, and we find $\tilde{f} = \tilde{\gamma} = 0$ and $\tilde{D} = \int_0^\infty d\tau\, \nu(\tau) = \int_0^\infty d\tau \int_0^\infty d\omega J(\omega)\cos(\omega\tau)$ (for $T = 0$). Using similar arguments about repeated cosine transform as we did below Eq. (4.56), we finally can write $\tilde{D} = \frac{\pi}{2}J(0)$. Since e.g. for the Ohmic spectral density this would be zero, let's rather take it as "lowest frequency contributions to the spectral density".

The Eq. (4.61) becomes

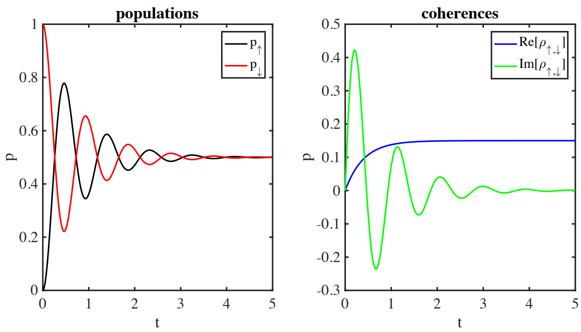
$$\frac{d}{dt}\hat{\rho}_{\mathcal{S}}(t) = -i\big[\hat{H}_{\mathcal{S}}, \hat{\rho}_{\mathcal{S}}(t)\big] - \tilde{D}\big[\hat{\sigma}_z, \big[\hat{\sigma}_z, \hat{\rho}_{\mathcal{S}}(t)\big]\big]. \tag{4.67}$$

For $\hat{H}_{\mathcal{S}} = 0$, we an write the entire (4.67) in 2×2 matrix notation, and find:

$$\frac{d}{dt}\begin{bmatrix} \rho_{\uparrow\uparrow}(t) & \rho_{\uparrow\downarrow}(t) \\ \rho_{\downarrow\uparrow}(t) & \rho_{\downarrow\downarrow}(t) \end{bmatrix} = -4\tilde{D}\begin{bmatrix} 0 & \rho_{\uparrow\downarrow}(t) \\ \rho_{\downarrow\uparrow}(t) & 0 \end{bmatrix}, \tag{4.68}$$

which immediately yields $\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0)\exp[-\tilde{D}t]$. We thus see that that coherences are exponentially damped with rate $\tilde{D}$. We can interpret this decoherence as arising because the environment "measures" the observable $\hat{S}_z$ via dynamics as shown in the example of section 2.2.1. Thus coherences in the basis $|\uparrow\rangle$, $|\downarrow\rangle$ are exponentially damped.

Numerical solution of Spin-Boson ME: without any simplifications. We can use a matrix representation $\rho_{\mathcal{S},nm}$ for the reduced density matrix $\hat{\rho}_{\mathcal{S}}(t) = \sum_{nm}\rho_{\mathcal{S},nm}(t)|\,n\,\rangle\langle\,m\,|$ of the spin, where $|\,n\,\rangle$ and $|\,m\,\rangle \in \{|\uparrow\rangle, |\downarrow\rangle\}$, and derive equations of motion $\frac{\partial}{\partial t}\rho_{\mathcal{S},nm}(t) = \cdots$ from (4.61) (assignment). These can then be solved without any further simplifications.



left: Results of Eq. (4.61) for $\Omega_0 = (2\pi)$, $\tilde{D} = (2\pi)/10$ and $\zeta = (2\pi)(0.05 + i0.03)$. We see initially coherent spin-oscillations due to $\Omega_0 > 0$, which are progressively decohering, governed by $4\tilde{D}$.

We can do the same explicit expansion for the remaining terms in (4.61) and find:

$$\frac{d}{dt}\begin{bmatrix} \rho_{\uparrow\uparrow}(t) & \rho_{\uparrow\downarrow}(t) \\ \rho_{\downarrow\uparrow}(t) & \rho_{\downarrow\downarrow}(t) \end{bmatrix} = \cdots + \begin{bmatrix} 0 & -2i[\zeta\rho_{\uparrow\uparrow}(t) + \zeta^*\rho_{\downarrow\downarrow}(t)] \\ -2i[\zeta\rho_{\downarrow\downarrow}(t) + \zeta^*\rho_{\uparrow\uparrow}(t)] & 0 \end{bmatrix}, \tag{4.69}$$

where $\cdots$ stands for pieces already discussed earlier. The effect of these terms is less obvious, they may even counteract decoherence.

4.6 Spontaneous decay

Now let us shift our attention to the two level atom interacting with a quantized photon field, that we had introduced in section 2.2.2. Since each photon mode is mathematically equivalent to a harmonic oscillator, and our two-level atom is equivalent to a spin-1/2 system, this falls into the category of system: spin - environment: oscillator.

We had already decomposed the Hamiltonian appropriately in (2.17)-(2.19). We do a slight modification (see changed version) and then let us only cast the interaction Hamiltonian into our usual form ($\hbar = 1$):

$$\hat{H}_{\text{int}} = \underbrace{\sum_{n\nu}\left(g_{n\nu}\hat{a}_{n\nu} + g^*_{n\nu}\hat{a}^\dagger_{n\nu}\right)}_{=\hat{E}}\underbrace{(\hat{\sigma}_+ + \hat{\sigma}_-)}_{=\hat{S}} \tag{4.70}$$

We follow the usual steps to obtain a Born-Markov ME, so we first require the environmental correlation function. We shall assume the photon-field to be in the thermal state (4.27), which can describe the vacuum for $T \to 0$ and otherwise incorporates black-body radiation at temperature T.

As before, the ladder operators in the interaction picture are simply $\hat{a}^{(I)}_{n\nu}(t) = \hat{a}_{n\nu}(0)e^{-i\omega_{n\nu}t}$. This yields the correlation function

$$\begin{aligned}
\mathcal{C}(\tau) = \langle\hat{E}^{(I)}(\tau)\hat{E}^{(I)}(0)\rangle &= \sum_{n\nu,n'\nu'}\langle\left(g_{n\nu}\hat{a}_{n\nu}(0)e^{-i\omega_{n\nu}\tau} + g^*_{n\nu}\hat{a}^\dagger_{n\nu}(0)e^{i\omega_{n\nu}\tau}\right)\left(g_{n'\nu'}\hat{a}(0)_{n'\nu'} + g^*_{n'\nu'}\hat{a}^\dagger_{n'\nu'}(0)\right)\rangle \\
&= \sum_{n\nu}|g_{n\nu}|^2\left(\langle\hat{a}_{n\nu}(0)\hat{a}^\dagger_{n\nu}(0)\rangle e^{-i\omega_{n\nu}\tau} + \langle\hat{a}^\dagger_{n\nu}(0)\hat{a}_{n\nu}(0)\rangle e^{i\omega_{n\nu}\tau}\right) \\
&= \sum_{n\nu}|g_{n\nu}|^2\left([N_{n\nu}(T)+1]e^{-i\omega_{n\nu}\tau} + N_{n\nu}(T)e^{i\omega_{n\nu}\tau}\right).
\end{aligned} \tag{4.71}$$

For the second line we used that photon modes with $n\nu \neq n'\nu'$ are uncorrelated in state (4.27), and that it is a mixture of number states. For the last line we have again made use of the thermal population $N_{n\nu}(T)$ of mode $n\nu$ at temperature T.

Next also we require the interaction picture evolution of the system operators, and find:

$$i\frac{\partial}{\partial t}\hat{\sigma}_\pm = \left[\hat{\sigma}_\pm, \hat{H}_{\mathcal{S}}\right] = \mp\omega_{ge}\hat{\sigma}_\pm, \tag{4.72}$$

and hence $\hat{S}^{(I)}(t) = \hat{\sigma}_+(0)e^{-i\omega_{ge}t} + \hat{\sigma}_-(0)e^{+i\omega_{ge}t}$.

In a final step, we combine the correlation function and $\hat{S}^{(I)}(t)$ into decoherence operators $\hat{B}$, $\hat{C}$ and find:

$$\begin{aligned}
\hat{B} &= \int_0^\infty d\tau\, \mathcal{C}(\tau)\hat{S}^{(I)}(-\tau) \\
&= \int_0^\infty d\tau\, \sum_{n\nu}|g_{n\nu}|^2\left([N_{n\nu}(T)+1]e^{-i\omega_{n\nu}\tau} + N_{n\nu}(T)e^{i\omega_{n\nu}\tau}\right) \times \left(\hat{\sigma}_+(0)e^{+i\omega_{ge}\tau} + \hat{\sigma}_-(0)e^{-i\omega_{ge}\tau}\right)
\end{aligned} \tag{4.73}$$

We now use the

Cauchy principal value:

$$\int_0^\infty d\tau\, e^{-i\omega\tau} = \pi\delta(\omega) + i\mathcal{P}\left(\frac{1}{\omega}\right). \tag{4.74}$$

This expression has to be thought of being applied onto a test function $f(\omega)$ and then integrated over, then $\int_{-\infty}^{\infty} d\omega\mathcal{P}\left(\frac{1}{\omega}\right) f(\omega) \equiv \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} d\omega\frac{f(\omega)}{\omega} + \int_{\epsilon}^{\infty} d\omega\frac{f(\omega)}{\omega}\right)$

We also convert $\sum_{n\nu} \to \int_0^\infty d\omega\rho(\omega)$, where $\rho(\omega)$ is the photon density of states [i.e. the number of photon modes in a small frequency interval $[\omega, \omega + d\omega]$]. Ignoring the principal value part for the moment, we then arrive at

$$\begin{aligned}\hat{B} &= \int_0^\infty d\tau\, \mathcal{C}(\tau)\hat{S}^{(I)}(-\tau)\\ &= \pi\int_0^\infty d\omega\rho(\omega)|g(\omega)|^2\left([N_\omega(T)+1]\hat{\sigma}_-\delta(\omega-\omega_{eg}) + N_\omega(T)\hat{\sigma}_+\delta(\omega-\omega_{eg})\right) + \mathcal{P}\text{ part}\\ &= \underbrace{\pi\rho(\omega_{eg})|g(\omega_{eg})|^2}_{\equiv\gamma/2}\left([N_{\omega_{eg}}(T)+1]\hat{\sigma}_- + N_{\omega_{eg}}(T)\hat{\sigma}_+\right) + \mathcal{P}\text{ part}\end{aligned} \tag{4.75}$$

In the second line we have already discarded two terms containing a delta function like $\delta(\omega + \omega_{ge})$ that cannot be fulfilled since all frequencies are positive, and in the third applied the remaining delta-functions. Similarly we find:

$$\hat{C} = \frac{\gamma}{2}\left([N_{\omega_{eg}}(T)+1]\hat{\sigma}_+ + N_{\omega_{eg}}(T)\hat{\sigma}_-\right) + \mathcal{P}\text{ part} \tag{4.76}$$

We finally insert (4.75) and (4.76) into (4.21), calculate lots of commutators and re-arrange. After the dust settles, we have the

Master equation for a two-level atom in a quantum radiation field

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{\mathcal{S}}(t) = -i\big[\hat{H}'_{\mathcal{S}}, \hat{\rho}_{\mathcal{S}}(t)\big] &+ \frac{\gamma}{2}[N_{\omega_{eg}}(T)+1]\Big(2\hat{\sigma}_-\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_{\mathcal{S}}(t) - \hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_+\hat{\sigma}_-\Big)\\ &+ \frac{\gamma}{2}N_{\omega_{eg}}(T)\Big(2\hat{\sigma}_+\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_{\mathcal{S}}(t) - \hat{\rho}_{\mathcal{S}}(t)\,\hat{\sigma}_-\hat{\sigma}_+\Big)\end{aligned} \tag{4.77}$$

$\hat{H}'_{\mathcal{S}} = \left(\frac{\hbar\omega_{eg}}{2} + \Delta E\right)\sigma_z$, where ΔE is a Lamb-shift from the $\mathcal{P}$ parts glossed over above.

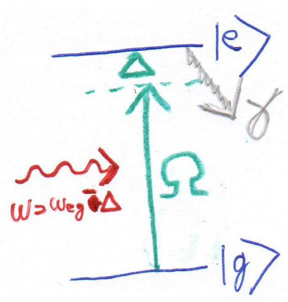
- Eq. (4.77) is of the Lindblad form (4.24), with two operators $\hat{L}_\mu \in \{\hat{\sigma}_+, \hat{\sigma}_-\}$.
- The first line, where $\hat{L} = \hat{\sigma}_-$, describes spontaneous decay and stimulated emission of the atom. The term is non-zero even in vacuum $T = 0$, and detailed inspection of Eq. (4.77)

(in the example below), reveals that this part redistributes population from the upper to the lower state.

- Hence the second line, where $\hat{L} = \hat{\sigma}_+$ is due to absorption of the incoherent black-body radiation.

- In Eq. (4.77) only the radiation density exactly on resonance affects the atom. This cannot be the full story and is an artefact of the Markov approximation used.

Example I: Decoherence of Rabi oscillations / Optical Bloch equations:



left: Consider an additional coherent laser field driving transitions from $|g\rangle$ to $|e\rangle$ as shown on the left. It turns out (PHY402), this can be described by an effective system (atom) Hamiltonian

$$\hat{H}_{\mathcal{S}} = \frac{\Omega}{2}\hat{\sigma}_x - \frac{\Delta}{2}\hat{\sigma}_z. \tag{4.78}$$

Since the atom can spontaneously decay while being illuminated by the laser, we solve (4.77) for $T = 0$ using[a] the Hamiltonian (4.80).

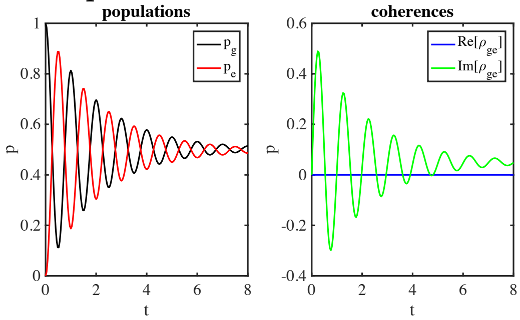
[a] Technically, since we changed the system Hamiltonian, we also ought to re-derive the ME, since the system operator evolution (4.72) will be different now. We don't, since spontaneous decay involves the energy/time scale defined by $\omega_{ge} \sim 500$THz, which is much larger/faster than Ω, $\Delta \sim$ MHz, GHz. So we can first find the effect of spontaneous decay as discussed in section 4.6, and then add the slow system evolution later.

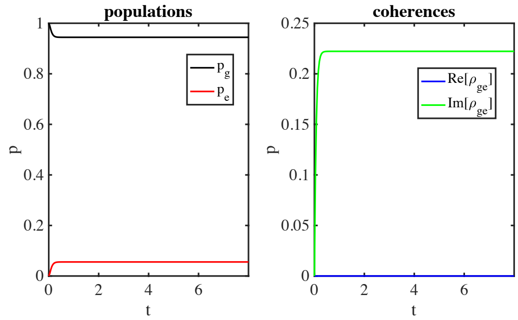
Written explicitly in terms of elements of the density matrix, we then find:

$$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix}\rho_{gg} & \rho_{ge}\\ \rho_{eg} & \rho_{ee}\end{bmatrix} &= \begin{bmatrix} i\frac{\Omega}{2}(\rho_{ge}-\rho_{eg}) & i\frac{\Omega}{2}(\rho_{gg}-\rho_{ee}) - i\Delta\rho_{ge}\\ i\frac{\Omega}{2}(\rho_{ee}-\rho_{gg}) + i\Delta\rho_{eg} & -i\frac{\Omega}{2}(\rho_{ge}-\rho_{eg})\end{bmatrix}\\ &+ \gamma[N_{\omega_{eg}}(T)+1]\begin{bmatrix}\rho_{ee} & -\frac{1}{2}\rho_{ge}\\ -\frac{1}{2}\rho_{eg} & -\rho_{ee}\end{bmatrix} + \gamma N_{\omega_{eg}}(T)\begin{bmatrix}-\rho_{gg} & -\frac{1}{2}\rho_{ge}\\ -\frac{1}{2}\rho_{eg} & \rho_{gg}\end{bmatrix}\end{aligned} \tag{4.79}$$

On the rhs, we have separated off the unitary part from $-i\big[\hat{H}_{\mathcal{S}}, \hat{\rho}_{\mathcal{S}}(t)\big]$ from the part related to spontaneous decay. We see that (4.81) can simultaneously describe coherent driving and incoherent decay of the atom, as well as incoherent excitation by BBR. The second line corroborates our interpretation given earlier. For $T = 0$, (4.81) are called the optical Bloch equations.

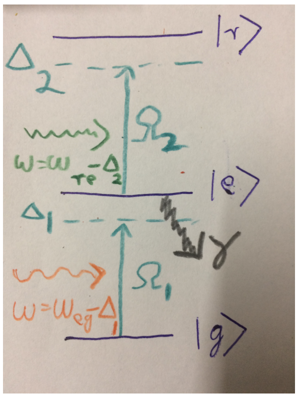
Example I continued:





Two exemplary numerical solutions of (4.81) for $\Delta = 0$ and $N_{\omega_{eg}}(T) = 0$ are shown on the top. The left one has $\Omega = (2\pi)$, $\gamma = (2\pi)/10$, so Rabi oscillations are seen but decohere on a time-scale $2\pi/\gamma$. The right one is for $\Omega = (2\pi)$, $\gamma = 4(2\pi)$, where no oscillations are visible but some equilibrium is reached quickly.

Example II: Coherent versus incoherent two-photon transition to Rydberg states:



left: Now we extend the earlier example by a second follow-up laser transition from the excited state $|\,e\,\rangle$ to an even higher excited state $|\,r\,\rangle$. If $|\,r\,\rangle$ is a Rydberg state (e.g. principal quantum number $n = 80$, see PHY 402), it makes sense to assume that $|\,r\,\rangle$ does *not* spontaneously decay, only $|\,e\,\rangle$ does, as shown in the figure.

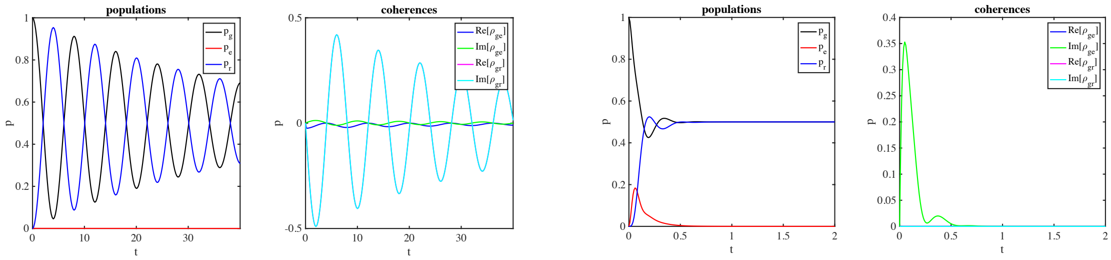
The effective system (atom) Hamiltonian for this case is

$$\hat{H}_{\mathcal{S}} = \frac{\Omega_1}{2}(|\,e\,\rangle\langle\,g\,| + |\,g\,\rangle\langle\,e\,|) - \Delta_1|\,e\,\rangle\langle\,e\,|$$
$$+ \frac{\Omega_2}{2}(|\,r\,\rangle\langle\,e\,| + |\,e\,\rangle\langle\,r\,|) - (\Delta_1 + \Delta_2)|\,r\,\rangle\langle\,r\,|. \tag{4.80}$$

If we include the decay of the middle level $|\,e\,\rangle$ (we now take $N(T) = 0$, i.e. $T = 0$) via the derived Lindblad operator $\hat{\sigma}_- \to |\,g\,\rangle\langle\,e\,|$, we find the three-level optical Bloch equations:

$$\dot{\rho}_{gg} = \gamma\rho_{ee} + i\frac{\Omega_1}{2}(\rho_{ge} - \rho_{eg}),$$
$$\dot{\rho}_{ee} = -\gamma\rho_{ee} - i\frac{\Omega_1}{2}(\rho_{ge} - \rho_{eg}) - i\frac{\Omega_2}{2}(\rho_{re} - \rho_{er}),$$
$$\dot{\rho}_{rr} = +i\frac{\Omega_2}{2}(\rho_{re} - \rho_{er}),$$
$$\dot{\rho}_{ge} = -\frac{\gamma}{2}\rho_{ge} + i\frac{\Omega_1}{2}(\rho_{gg} - \rho_{ee}) + i\frac{\Omega_2}{2}\rho_{gr} - i\Delta_1\rho_{ge},$$
$$\dot{\rho}_{gr} = -i\frac{\Omega_1}{2}\rho_{er} + i\frac{\Omega_2}{2}\rho_{ge} - i(\Delta_1 + \Delta_2)\rho_{gr},$$
$$\dot{\rho}_{er} = -\frac{\gamma}{2}\rho_{er} - i\frac{\Omega_1}{2}\rho_{gr} - i\frac{\Omega_2}{2}(\rho_{rr} - \rho_{ee}) - i\Delta_2\rho_{er}. \tag{4.81}$$

Example II continued:
As before we can solve these numerically:



(left two panels) Populations and coherences for $\Omega_1 = \Omega_2 = (2\pi)5$ and $\Delta_1 = -\Delta_2 = (2\pi)100$. Hence $\Delta_1 + \Delta_2 = 0$ and the *two-photon* transition is resonant for $|g\rangle \leftrightarrow |r\rangle$. However since $|\Delta_k| \gg |\Omega_k|$, we say that this transitions is proceeding *off-resonantly* via $|e\rangle$. Thus population in $|e\rangle$ and thus spontaneous decay and decoherence can be kept small. This is used in experiments in practice to create coherent transitions between ground- and Rydberg states using two lasers.
(right two panels) Populations and coherences for $\Omega_1 = \Omega_2 = (2\pi)5$ and $\Delta_1 = \Delta_2 = 0$. Now we proceed *resonantly* via the middle level. As a result excitation is strongly decohered by spontaneous decay.

Example III: Adiabatic Elimination:
Let us understand the coherent coupling in example II using adiabatic elimination. In the equations for $\dot{\rho}_{ge}$ the by far largest term on the rhs is $-i\Delta_1\rho_{ge}$. This will cause the complex number ρ_{ge} to very quickly rotate. If we coarse grain in time, this allows us to actually set $\dot{\rho}_{ge} = 0$. *Warning: The implication is confusingly not that it varies too slowly, but rather too fast.* We can then solve the resultant *algebraic equation*, using also $\Delta_1 \gg \gamma$ to yield:

$$\rho_{ge} \approx \frac{\Omega_1}{2\Delta_1}(\rho_{gg} - \rho_{ee}) + \frac{\Omega_2}{2\Delta_1}\rho_{gr}. \tag{4.82}$$

Similarly for $\dot{\rho}_{er}$ we find

$$\rho_{er} \approx -\frac{\Omega_2}{2\Delta_2}(\rho_{rr} - \rho_{ee}) - \frac{\Omega_1}{2\Delta_2}\rho_{gr}. \tag{4.83}$$

Inserting these into the remaining equations gives (assuming $\rho_{ee} \approx 0$)

$$\begin{aligned}
\dot{\rho}_{gg} &= +i\frac{\Omega_{\text{eff}}}{2}(\rho_{gr} - \rho_{rg}), \\
\dot{\rho}_{rr} &= +i\frac{\Omega_{\text{eff}}}{2}(\rho_{rg} - \rho_{gr}), \\
\dot{\rho}_{gr} &= i\frac{\Omega_{\text{eff}}}{2}(\rho_{gg} - \rho_{rr}),
\end{aligned} \tag{4.84}$$

where $\Omega_{\text{eff}} = \frac{\Omega_1\Omega_2}{2\Delta_1}$ takes the place of an *effective Rabi frequency* of our coherent two-photon transition. The corresponding Rabi period is $T_{\text{rab}} = 2\pi/\Omega_{\text{eff}} = 8$ for example II, matching the observation in the left-most panel.

4.7 Steady states of a Master equation

The second example above shows a phenomenon that frequently happens when dealing with Master equations: At some time the dissipative or decohering terms have established a steady state, where none of the density matrix element change any more. Often this is of major interest, particularly if the time it takes to establish it (called the "transient") is too short to be interesting for us. Mathematically we can define a

Steady state of a Master equation simply by demanding

$$\frac{d}{dt}\hat{\rho}(t) = 0, \text{ and } \mathrm{Tr}[\hat{\rho}^{(ss)}] = 1. \tag{4.85}$$

We call the solution of this $\hat{\rho}^{(ss)}$, for "steady state".

This can often be solved much easier than the actual ME, since it is just an algebraic equation.

Example: Steady state of the driven atom:
Let us apply the concept to the example earlier. We combine Eq. (4.85) with Eq. (4.81), hence we just set the lhs of Eq. (4.81) to zero and solve the resultant system of algebraic equations including $\rho_{gg}^{(ss)} + \rho_{ee}^{(ss)} = 1$. We find (for $\Delta = 0$)

$$\begin{aligned} \rho_{gg}^{(ss)} &= \frac{\gamma^2 + \Omega^2}{\gamma^2 + 2\Omega^2}, & \rho_{ee}^{(ss)} &= \frac{\Omega^2}{\gamma^2 + 2\Omega^2}, \\ \rho_{ge}^{(ss)} &= \frac{i\gamma\Omega}{\gamma^2 + 2\Omega^2}, & \rho_{eg}^{(ss)} &= \frac{-i\gamma\Omega}{\gamma^2 + 2\Omega^2}. \end{aligned} \tag{4.86}$$

These values match with the steady simulation results found in example-I earlier.
The same could be applied to example-II, right panels, to find the equilibrium populations of all the levels (exercise).