Phys635, MBQM II-Semester 2022/23, Tutorial 6 solution

Stage 1 Following the procedure of the lecture, calculate the energy of hole or particle excitations in a Fermi liquid (homogeneous system as in lecture) to first order perturbation theory. Then take their energy relative to the unperturbed Fermi sea. Make a graph of particle/hole energy as a function of k and discuss your results.

Solution: The Hamiltonian in momentum space is (see Eq. (4.24) lecture),

$$\hat{H} = \sum_{\mathbf{k},s} \frac{\hbar^2 \mathbf{k}^2}{2m} \hat{a}^\dagger_{\mathbf{k},s} \hat{a}_{\mathbf{k},s} + \frac{U_0}{\mathcal{V}} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \hat{a}^\dagger_{\mathbf{k}_3,\uparrow} \hat{a}^\dagger_{\mathbf{k}_4,\downarrow} \hat{a}_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_1,\uparrow} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} \tag{1}$$

where $s \in \{\uparrow, \downarrow\}$. We do perturbation theory in the interaction term, let's call it $\hat{V}$.

For a particle or hole excitation the unperturbed energy is $E^{(0)}_k = \hbar^2(k^2 - k_f^2)/2/m$ (see below Eq. 4.22). The first order energy correction is

$$\Delta E^{(1)} = \langle\, (\mathbf{k}^p \uparrow) \,|\hat{V}|\, (\mathbf{k}^p \uparrow) \,\rangle, \tag{2}$$

where we used the notation (4.28) and specified a particle state with spin-up for the moment.

Thus

$$\begin{aligned}
\Delta E^{(1)} &= \frac{U_0}{\mathcal{V}} \langle\, (\mathbf{k}^p \uparrow) \,| \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \hat{a}^\dagger_{\mathbf{k}_3,\uparrow} \hat{a}^\dagger_{\mathbf{k}_4,\downarrow} \hat{a}_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_1,\uparrow} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} |\, (\mathbf{k}^p \uparrow) \,\rangle \\
&= \frac{U_0}{\mathcal{V}} \langle\, (\mathbf{k}^p \uparrow) \,| \sum_{|\mathbf{k}_1|<k_f,\{|\mathbf{k}_2|<k_f \text{ or } \mathbf{k}_2=\mathbf{k}_p\},\mathbf{k}_3=\mathbf{k}_1,\mathbf{k}_4=\mathbf{k}_2} \hat{a}^\dagger_{\mathbf{k}_3,\uparrow} \hat{a}^\dagger_{\mathbf{k}_4,\downarrow} \hat{a}_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_1,\uparrow} |\, (\mathbf{k}^p \uparrow) \,\rangle,
\end{aligned} \tag{3}$$

The constraints on the sum has come about as follows: The destruction operators $\hat{a}_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_1,\uparrow}$ have to act on a filled state, thus within the Fermi sea, or onto the particle excitation. Then the creation operators have to act on exactly the same single body state, for the result (rhs. ket after acting with all operators onto it) to be non-orthogonal to the bra. The quantum state is a product of Fock states for spin-up and for spin-down, which allows us to write (with using $k_3 = k_1$, $k_4 = k_2$):

$$\begin{aligned}
\Delta E^{(1)} &= \frac{U_0}{\mathcal{V}} \sum_{|\mathbf{k}_1|<k_f} \sum_{\{|\mathbf{k}_2|<k_f \text{ or } \mathbf{k}_2=\mathbf{k}_p\}} \langle\, (\mathbf{k}^p \uparrow) \,|\hat{a}^\dagger_{\mathbf{k}_1,\uparrow} \hat{a}^\dagger_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_1,\uparrow}|\, (\mathbf{k}^p \uparrow) \,\rangle, \\
&= \frac{U_0}{\mathcal{V}} \sum_{|\mathbf{k}_1|<k_f} \sum_{\{|\mathbf{k}_2|<k_f \text{ or } \mathbf{k}_2=\mathbf{k}_p\}} \langle\, (\mathbf{k}^p \uparrow) \,|\hat{a}^\dagger_{\mathbf{k}_2,\downarrow} \hat{a}_{\mathbf{k}_2,\downarrow}|\, (\mathbf{k}^p \uparrow) \,\rangle \langle\, (\mathbf{k}^p \uparrow) \,|\hat{a}^\dagger_{\mathbf{k}_1,\uparrow} \hat{a}_{\mathbf{k}_1,\uparrow}|\, (\mathbf{k}^p \uparrow) \,\rangle,
\end{aligned} \tag{4}$$

In the second line we have used that operators for opposite spin states anti-commute and the state is a simple tensor product $|\,(\mathbf{k}^p \uparrow)\,\rangle = |\,\mathbf{N}_\uparrow\,\rangle \otimes |\,\mathbf{N}_\downarrow\,\rangle$. Here $\mathbf{N}_\uparrow$ is a vector of occupation numbers for all spin up particles (which includes the excitation) and $\mathbf{N}_\downarrow$ one for the spin down particles (with no excitation, only filled Fermi sea). This tensor product structure allows us to factor the average into one pertaining to spin-up and one to spin-down.

We can write the last line simply as $\Delta E^{(1)} = U_0 N_\uparrow N_\downarrow / \mathcal{V}$, exactly as for the energy shift of the Fermi sea itself (see Eq. (4.26)).

If we always compare particles and holes with a Fermi sea of equal particle numbers, we thus get the same dispersion relation as without interactions: $E_k = \hbar^2(k^2 - k_f^2)/2/m$.

Stage 2 Cooper pairs

(i) Understand and discuss the cartoon picture of the cooper pairing mechanism given in the lecture. *See discussion in lecture: green box, page 100.*

(ii) Understand and discuss the two-body cooper pair calculation given in the lecture. *See discussion in lecture: section 4.10.1.*

Stage 3 If we covered it in the lecture: discuss the following:

(i) Why are attractively interacting degenerate Fermions fundamentally different from repulsive ones? *See discussion in lecture: Attractive interactions give rise to pairing (see below) and thus make the simple filled Fermi sea a bad starting point for perturbation theory.*

(ii) What is a cooper pair?
It is the bound-state of two opposite spin Fermions in the presence of a Fermi-sea blocking momenta up to k_F, which can then arise due to arbitrarily weak interactions. In solids, two electrons can pair up by effective attractive interactions mediated by lattice phonons. These can have ranges much larger than the screened Coulomb repulsion.

(iii) What is the BCS state?
The complex many-body state that we obtain when translating the cooper pair concept to all-vs-all interactions. Electrons with opposite spins and moments are paired up (or not) all the way up the Fermi surface. See lecture.

(iv) Why/when are degenerate Fermi gases superfluid? *The spectrum has a gap for non-zero pairing field Δ, so in that case they become superfluid.*