Phys635, MBQM II-Semester 2022/23, Tutorial 5 solution

Stage 1 Fermi energy

(i) Can you think of electrons in an atom as degenerate Fermi gas? When do you expect this to be useful, when not so much?
Solution: The attraction by the nucleus crams all electrons into a small spatial volume, for low lying states usually comparable in spatial scale to their de-Broglie wavelength (roughly distance between nodes in wavefunctions). Thus they do form a degenerate gas. You could also just say that you know that the Pauli exclusion principle is crucial for the structure of atoms. However for small atoms with, say 1-5 electrons, we would expect the form and energies of electronic states to govern everything. For larger atoms it might make some sense to think about a total local electron density instead, and use methods from the present lecture. See also Q1 in assignment 6.

(ii) Let us just assume we can use the concept. In the AMO lecture you would have learnt that the energy of an electron in a Hydrogenic atom (single electron, nuclear charge Z) is

$$E_n = -\frac{m_e e^4}{8\epsilon_0^2 h^2}\frac{Z^2}{n^2} = -13.6\text{eV}\frac{Z^2}{n^2} \tag{1}$$

where n is the principal quantum number. Find the Fermi-energy of Argon ($Z = 18$), assuming electrons do not interact.
Solution: If electrons do not interact we can use the above formula for all levels. To fill the 18 electron in argon, we have to go up to $n = 3$ *(see e.g. https://ptable.com), which are all generate (only because we neglect electron interactions). The most energetic Fermion thus sits at* $E_3 = -13.6eV\frac{(18)^2}{3^2} = -489.6eV$ *and the lowest one at* $E_1 = -13.6eV\frac{(18)^2}{1^2} = -4406.4eV$*. We take the Fermi energy as the energy of the highest occupied state relative to the lowest one, thus* $E_F = 3916.8$ eV*.*

Stage 2 Fermi pressure: Similar to how we found the 3D Fermi pressure in the lecture, find it in 2D and 1D. Compare the three cases and discuss.
Solution: The method is the same, just that the sum and subsequent integrals over states in the lines above Eq. (4.10) are now in lower dimension. You should find

(disregarding spin degeneracy) Fermi energies

$$E_{F,1D} = \frac{\hbar^2}{2m}\left(\pi\rho_{1D}\right)^2, \tag{2}$$

$$E_{F,2D} = \frac{\hbar^2}{2m}\left(4\pi\rho_{2D}\right), \tag{3}$$

$$E_{F,3D} = \frac{\hbar^2}{2m}\left(6\pi^2\rho_{3D}\right)^{2/3}. \tag{4}$$

and from that degeneracy pressures:

$$P_{F,1D} = \frac{2}{3}\rho_{1D}E_{F,1D}, \tag{5}$$

$$P_{F,2D} = \frac{1}{2}\rho_{2D}E_{F,2D}, \tag{6}$$

$$P_{F,3D} = \frac{2}{5}\rho_{3D}E_{F,3D}. \tag{7}$$

We find a steeper scaling with (low-dimensional) density for lower numbers of dimensions, which we can interpret as due to the fact that in lower dimensions, there are fewer states to distribute the Fermions over, up to a given energy.

Stage 3 Interacting Fermions:

(i) How does statistics affect atomic scattering? Which type of particles (Bosons or Fermions) has which partial waves?
Solution: If we consider the symmetry or anti-symmetry requirements for the different terms of the partial wave expansion, see near Eq. (4.18), we discover that identical *Bosons can have only even terms and Fermions only odd terms in the PWE. Since the arguments that low-temperature scattering of dilute gases is dominated by s-wave scattering still hold, we conclude that the interactions between identical Fermions are strongly suppressed at low enough temperatures.*

(ii) What is a Fermi sea?
Solution: It is the quantum state with all states up to the Fermi energy filled with exactly one Fermion, see Eq. (4.19).

(iii) What is a Fermi liquid?
see discussion in lecture and literature

(iv) What are the excitations of a Fermi liquid?
Particles and hole excitations, for a non-interacting system. see discussion in lecture 4.8.2. In the presence of interactions, these turn into dressed particle and dressed hole excitations, see section 4.9.

(v) What is many-body dressing?
see Eq. (4.40) and image on page 90. In quantum mechanics, we generally call a state $|\,\phi\,\rangle = \sqrt{1-\epsilon}|\,a\,\rangle + \sqrt{\epsilon}|\,b\,\rangle$*, to be "a state where* $|\,a\,\rangle$ *is* <u>*dressed with*</u> $|\,b\,\rangle$*". If* $|\,\phi\,\rangle$*,* $|\,a\,\rangle$*,* $|\,b\,\rangle$ *are all many-body states, we refer to this as many-body dressing.*

(vi) Make your own cartoon drawing of the ground state of an interacting Fermi liquid. How important is each contribution/basis state?
Solution: See images on page 96 and 97.