Phys635, MBQM II-Semester 2022/23, Tutorial 4 solution

Stage 1 Condensate: ground-states and dynamics

(i) Consider a Bose-Einstein condensate in the double well potential below for the given chemical potential. Discuss how you would expect the ground-state density to look like, qualitatively. Which cases should we distinguish?
Solution: The ground state density should look like the red line below, as long as the healing length ξ *corresponding to the peak density is small compared to* d *and* a*. If they are, we can use the TF approximation for most of the range of* x*, but have to be careful at discontinuities, where we use a pictures as in section 3.3.5. (derivation of healing length).*

In contrast, for much smaller nonlinearities, it might look like the single particle solution of this potential, (I guessed, the blue line). If you want, you can try to check and hopefully confirm all this using the imaginary time code from Assignment 4 Q3.

(ii) Starting from this ground-state, if one now suddenly increases the potential V_L and decreases V_R what should happen in time and why? How can you know? When does it get complicated?
Hints: What do the hydrodynamic equations tell you about condensate behavior? We stated the ones following from the TDGPE. Can you guess the ones following from the TIGPE and use them here?

Solution: The equations tell us that the condensate behaves like a fluid if we neglect the quantum pressure term. We can re-interpret the trapping potential e.g. as gravitational potential, then the raising and lowering of the potential corresponds to raising and lowering buckets containing that fluid which are connected by a pipe. We would thus expect the fluid (condensate) to start flowing from the higher potential well to the lower one.

To see this in another way, we get the time-independent version of the Bernoulli equation (3.56) by setting time derivatives to zero:

$$0 = -\boldsymbol{\nabla}[P_q + \frac{1}{2}m\mathbf{v}^2 + U\rho + V(x)]] \Rightarrow$$
$$const = P_q + \frac{1}{2}m\mathbf{v}^2 + U\rho + V(x) \tag{1}$$

In the present steady state we need $\mathbf{v} = 0$ *and let us neglect* P_q *(assuming a Thomas-Fermi regime).*

Then prior to the change of potential we had $U\rho_L + V_L = U\rho_0 + V_0 = U\rho_R + V_R = const$*. If we now suddenly change* $V_{L/R}$ *as indicated, the condensate will, at least initially, attempt to equilibrate this relation again by lowering the density in the high potential reason and raising it in the* *low one.*

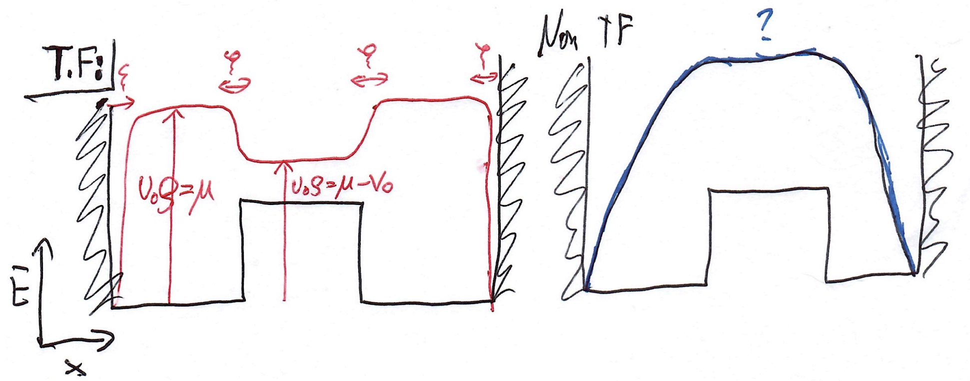

Abbildung 1: Double square well potential, $V(x) = \infty$ for $x < 0$ and $x > 2a+d$, $V(x) = V_L$ for $0 < x < a$, $V(x) = V_0$ for $a < x < a + d$, $V(x) = V_R$ for $a + d < x < 2a + d$ (see coordinates on question sheet). Solution: (left) Thomas Fermi regime, (right) Non TF regime (my educated guess, please implement imaginary time to know for sure).

Stage 2 Bogoliubov-excitations:

(i) What are "Bogoliubov-excitations"?
They are the quasi-particles of a BE condensed systems, which means that we can write the Hamiltonian in a diagonal form with only quasi-particle number operators, assuming that the density of quasi particles is low. If the Hamiltonian takes that form, that means that quasi-particles no-longer interact (which means that we have taken into account the most important effects of interactions correctly). Physically, in BEC, Bogoliubov-excitations are sound waves at large wavelengths (compared to the healing length) and just the excitation (momentum kick) of a single individual atom at small wave-length (or large energies/momenta)

(ii) What can we learn from them?
Their properties depend on the density and interaction strength, so we may learn that from e.g. finding a phonon-spectrum. BdG energies also govern whether a certain setup is stable.

(iii) Under which conditions can you learn about their time-evolution from the GPE?
If they are "macroscopically occupied". The same mode-shapes and energies arise when asking about the effect of a small perturbation of the condensate mean-field.

(iv) How would you create any?
Any sudden change of the Hamiltonian will create some, to be more controlled we want a small change only.

Stage 3 Condensate stability:

(i) Consider an infinitely extended 1D BEC in a homogenous initial state of density ρ with attractive interactions $U_0 < 0$. If we perturb this from perfect homogeneity with the perturbation having a wavelength (i) $\lambda = \hbar\pi/\sqrt{m|U_0|\rho}/4$, (ii) $\lambda = \hbar\pi/\sqrt{2m|U_0|\rho}$, (iii) $\lambda = 2\hbar\pi/\sqrt{m|U_0|\rho}$, which of these perturbations do you expect to remain small, which grow exponentially?

Solution: We look at the Bogoliubov dispersion relation, Eq. (3.71), which for $U_0 < 0$ is

$$\varepsilon_q \equiv \hbar\omega_q = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} - 2|U_0|\rho\right)} \tag{2}$$

The energy becomes imaginary for $\frac{\hbar^2 q^2}{2m} < 2|U_0|\rho$, corresponding to wavelengths $\lambda > \lambda_{crit} = \frac{\hbar\pi}{\sqrt{m|U_0|\rho}}$. These wavelength will thus be dynamically unstable (see section 3.4.6) and perturbations of this size grow exponentially. Thus (i) and (ii) will grow, while (iii) will remain small.

(ii) If the same BEC was contained in a large square well potential of size $a = \hbar\pi/\sqrt{2m|U_0|\rho}$, would you expect this to be stable or not?

Solution: In that case, there can be no modes with a wavelength large enough to be stable, and the system should remain stable.