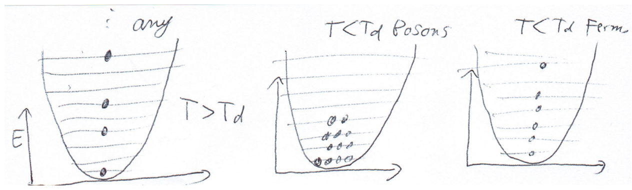
Phys635, MBQM II-Semester 2022/23, Tutorial 3 solution

Stage 1 Consider the attached two diagrams for Bosons or Fermions in a harmonic trap. The temperature where it becomes important whether two particles can enter the same state or not, is called "degeneracy temperature" T_d.



(i) Why would it be <u>not</u> important for some temperatures? *Solution: At large temperatures each particle has a high probability p to be in a fairly large energy state,* $p \sim \exp(-E/[k_B T])$. *Since this means that a large number of states are available for each particle, the chance of them "wanting to be" in the same single particle state is fairly small. In that case it also obviously doesn't matter whether they are allowed to be in that same state.*

(ii) Make a rough estimate of the degeneracy temperature. *Solution: States with* $E \sim k_B T$ *become likely. Thus we want* $k_B T \gg N\hbar\omega$ *in order for having a larger number of available states than particles.*

(iii) Suppose we are slowly reducing the temperature of the system. Discuss what might happen in the context of section 2.2.2 of the lecture. *Solution: At temperatures below the transition temperatures, if the system can equilibrate, more than one particle will " want to" get into the same state. For Fermions this will be forbidden due to Fermi blocking. For Bosons it will actually speed up, so once there are a few seed Boson in the groundstate, Bose-enhancement will <u>accelerate</u> the actual process of condensation. Importantly, like in 2.2.2., for having any transitions between harmonic oscillator states at all, we need additional terms in the Hamiltonian beyond the harmonic oscillator, for example interactions.*

Stage 2 Use a math plotting tool (such as mathematica) to explore Eq. (3.12). If you copy paste the following lines into mathematica, you can start comparing BE distributions $m(E_b)$ for two different sets of parameters.

```
kbT1 := 2;
kbT2 := 1;
mu1 := -2;
mu2 := -2;
Plot[{1/(Exp[(Eb -mu1)/kbT1] - 1), 1/(Exp[(Eb -mu2)/kbT2] - 1)}, {Eb, 0, 5}]
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(i) Let us assume a constant density of states $g(E)$ for simplicity. Confirm that when you reduce the temperature, the total number of particles $N = \int_0^\infty dEg(E)m(E_b)$ does down.
Solution: Let $T_1 > T_2$, you see that $m(E_b, T_1) > m(E_b, T_2)$ for <u>*all*</u> *values of E_b, hence you know the statement is correct without doing any integration.*

(ii) Suppose you want to keep the total number constant [while reducing the temperature] what do you do?
Since the only other control knob in the distribution function is μ, we have to adjust that one. It is negative, we have to increase it towards 0.

(iii) We cannot have $\mu > 0$. Can you find a way to keep the total number constant once $\mu = 0$ and you further reduce the temperature?
Not as before. As discussed in the lecture, the only way out is assuming the ground-state is macroscopically occupied (hence BEC).

Stage 3 Mean-field theory: Discuss:

(i) What is "mean-field" about mean-field theory?
A single particle would experience an interaction that depends on the positions of all the other particles in $|\psi(\mathbf{r}_1, \cdot\mathbf{r}_N)|^2$. If more of the others are close, interaction energy is higher else lower. Now the positions of other particles in turn might depend on each other, giving rise to <u>*correlations*</u>*. These are neglected in MFT, where the interaction is just calculated using the mean (probability) to find a particle in a certain place (e.g. at the point of the first atom). In GP theory, the Ansatz for ψ then also has no correlations, but the actual assumption in MFT is just that those are small/unimportant.*

(ii) What are the most important assumptions about the atomic gas for MFT to be valid?
Very low temperature, such that condensation happens and s-wave scattering is valid. Very dilute, such that interaction range $\ll$ interatomic spacing. Weak interactions.

(iii) What is the interpretation of the condensate wavefunction ?
It is the wavefunction ϕ shared by all the atoms such that $\psi(\mathbf{r}_1, \cdot\mathbf{r}_N) = \prod_k \phi(\mathbf{r}_k)$. This massively simplifies the many-body problem, since we only need to know about a 3D wave function, rather than a $3 \times N$ D one. Depending on taste, the condensate wave function maybe normalised to N instead of 1.

(iv) If you want to theoretically study a certain Bose-Einstein condensate experiment (after condensation has happened), which equations can you use? What do they tell you? What input information do you need?
We can use the TIGPE and TDGPE. The first one gives us the shape of the atomic density and momentum distributions. Its input information is the external potential felt by the atoms, their scattering length, mass and the number of atoms. The TDGPE then can tell us the response of the experiment to time-dependent changes such as changing the potential, or

inserting new ones, or changing the interactions with a Feshbach resonance (see week 12).