Phys635, MBQM II-Semester 2022/23,
Tutorial 1, Tue 17.1.

Please sit in your assignment teams, two or three teams to a table (behave like Bosons, not like Fermions). Do the "Stages" in the order below. When all teams finished a stage, elect a student to present and explain on the board.

Stage 1 Why is quantum-many-body theory more challenging than classical many-body theory? Discuss on the table, write on the board.

(i) What information is needed to specify a classical state of N particles? A quantum state?

(ii) Invent a way to "quantify" the volume of that information? How does either scale as the number of particles gets larger?

(iii) In terms of the classification of many-body states seen in the lecture, which aspect is "causing the trouble"?

Stage 2 Second quantisation:

(i) Show the commutation relations (2.8) from the definition of creation and destruction operators (2.4)-(2.7) using test Fock-states (2.2).

(ii) Show that the anti-symmetry of the Fermionic two-mode state $\langle x \,|\, 11 \rangle$ (see second dotpoint below Eq. (2.7)) under exchange of the two mode-labels a and b is correctly captured when using definition (2.6) and incorrectly when skipping the factor $(-1)^{\sum_{k<n} N_k}$.

(iii) Consider the Hamiltonian in second quantisation:

$$\hat{H} = \sum_m E_m \hat{a}_m^\dagger \hat{a}_m \tag{1}$$

where $\hat{a}_m$ destroy spin-1 bosons in a single (irrelevant) spatial mode and spin states $|\, s = 1, m_s = m \,\rangle$, $m = -1, 0, 1$, i.e. we are using eigenstates of $\hat{S}_z$ as single particle basis. What is the physical meaning of this Hamiltonian?

Now convert this Hamiltonian into one based on the single particle basis of eigenstates of $\hat{S}_x$, calling the corresponding operators $\hat{b}_m$, where $\hbar m$ is the eigenvalue of $\hat{S}_x$.

Stage 3 For the following first quantised Hamiltonians describing N particles, find the second quantized one that is equivalent. If there isn't any right option, find the right one.

(i) First quantized: $\hat{H} = -\sum_n^N \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_n^2} + \sum_{n,k;n\neq k} \frac{A}{2} e^{-\frac{(r_n-r_k)^2}{2S^2}}$, second quantized:

$$\hat{H}_1 = \int dk \int dk' \left(\frac{\hbar^2 kk'}{2m}\right) \hat{a}_k^\dagger \hat{a}_{k'} + A \int dk \int dk' O_{kk'} \hat{a}_k^\dagger \hat{a}_{k'},$$
$$\hat{H}_2 = \int dk \left(\frac{\hbar^2 k^2}{2m}\right) \hat{a}_k^\dagger \hat{a}_k + A \int dk \int dk' \int dq O_q \hat{a}_k^\dagger \hat{a}_{k'}^\dagger \hat{a}_{k'+q} \hat{a}_{k-q},$$
$$\hat{H}_3 = \int dk \left(\frac{\hbar^2 k^2}{2m}\right) \hat{a}_k^\dagger \hat{a}_k + A \int dk \int dk' \int dk'' \int dk''' O_{kk'} O_{k''k'''} \hat{a}_k^\dagger \hat{a}_{k'}^\dagger \hat{a}_{k''} \hat{a}_{k'''}, \tag{2}$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ create and destroy particles in states $|\,\phi_k\,\rangle$ with well defined momentum $p = \hbar k$ [also for (ii) and (iii) below], and $O_{kk'} = \frac{1}{2\pi}\int dx e^{i(k'-k)x} e^{-\frac{x^2}{2S^2}}$.

(ii) First quantized: $\hat{V} = \sum_{n,k,\ell;n\neq k\neq \ell} \frac{A}{3}\frac{1}{|r_n-r_k|^6}\left(\tanh\left[\frac{|(r_n+r_k)/2-r_\ell|+D}{\xi}\right]+1\right)$, second quantized:

$$\hat{V}_1 = A \int dk \int dk' O_{kk'} \hat{a}_k^\dagger \hat{a}_{k'},$$
$$\hat{V}_2 = A \int dk \int dk' \int dk'' \int dk''' O_{kk'k''k'''} \hat{a}_k^\dagger \hat{a}_{k'}^\dagger \hat{a}_{k''} \hat{a}_{k'''},$$
$$\hat{V}_3 = A \int dk_1 \int dk_2 \int dk_3 \int dk_4 \int dk_5 \int dk_6$$
$$O_{k_1,k_2,k_3,k_4,k_5,k_6} \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \hat{a}_{k_3}^\dagger \hat{a}_{k_4} \hat{a}_{k_5} \hat{a}_{k_6}. \tag{3}$$

where $O_{kk'} = \frac{1}{(2\pi)}\int dx e^{i(k'-k)x} A\frac{1}{x^6}\left(\tanh\left[\frac{|x/2-r_\ell|+D}{\xi}\right]+1\right)$,
$O_{kk'k''k'''} = \frac{1}{(2\pi)^2}\int dx \int dy e^{i(k''-k)x} e^{i(k'''-k')y} A\frac{1}{x^6}\left(\tanh\left[\frac{|x/2-y|+D}{\xi}\right]+1\right)$,
$O_{k_1,k_2,k_3,k_4,k_5,k_6} = \frac{1}{(2\pi)^3}\int dx \int dy \int dz e^{i(k_1-k_4)x} e^{i(k_2-k_5)y} e^{i(k_3-k_6)z} A\frac{1}{x^6}\left(\tanh\left[\frac{|(x+y)/2-z+D}{\xi}\right]+1\right)$

(iii) First quantized: $\hat{V} = \sum_{n,k;n\neq k} \frac{A}{2}\frac{1}{|r_n-r_k|^6}$, second quantized:

$$\hat{V}_1 = A \int dk \int dk' O_{kk'} \hat{a}_k^\dagger \hat{a}_{k'},$$
$$\hat{V}_2 = A \int dk \int dk' O_{kk'} \hat{a}_k^\dagger \hat{a}_{k'}^\dagger \hat{a}_{k'} \hat{a}_k,$$
$$\hat{V}_3 = A \int dk_1 \int dk_2 \int dk_3 \int dk_4 \int dk_5 \int dk_6$$
$$O_{k_1,k_2,k_3,k_4,k_5,k_6} \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \hat{a}_{k_3}^\dagger \hat{a}_{k_4} \hat{a}_{k_5} \hat{a}_{k_6}. \tag{4}$$

where $O_{kk'} = \frac{1}{(2\pi)}\int dx e^{i(k'-k)x} A\frac{1}{x^6}$,
$O_{k_1,k_2,k_3,k_4,k_5,k_6} = \frac{1}{(2\pi)^3}\int dx \int dy \int dz e^{i(k_1-k_4)x} e^{i(k_2-k_5)y} e^{i(k_3-k_6)z} A\frac{1}{|x-y|^6}$.