Phys635, MBQM II-Semester 2022/23, Tutorial 1 solution

Stage 1 Why is quantum-many-body theory more challenging than classical many-body theory? Discuss on the table, write on the board.

(i) What information is needed to specify a classical state of N particles? A quantum state? *Solution: For the classical state, we write down e.g. a phase space point* $[\mathbf{r}_1, \cdots \mathbf{r}_N; \mathbf{p}_1, \cdots \mathbf{p}_N]$. *In quantum theory a many-body wave-function* $\psi(\mathbf{r}_1, \cdots \mathbf{r}_N) \in \mathbb{C}$.

(ii) Invent a way to "quantify" the volume of that information? How does either scale as the number of particles gets larger? *Solution: Suppose we limit the number of available positions/momenta to M (or in QM the number of modes to M). Then the classical phase-space vector contains $2N$ real numbers. The QM wavefunction (Eq. (1.24)), contains M^N complex numbers.*

(iii) In terms of the classification of many-body states seen in the lecture, which aspect is "causing the trouble"? *Solution: Entanglement. If it wasn't for entanglement (i.e. we look at a separable state), the information contained again reduces to $M \times N$ complex numbers (why?), which is not so much worse than classical (we say it has the same scaling with N).*

Stage 2 Second quantisation:

(i) Show the commutation relations (2.8) from the definition of creation and destruction operators (2.4)-(2.7) using test Fock-states (2.2).
Solution: We apply the LHS of the commutation relation(s) to an arbitrary test Fock state. For Bosons:

$$(\hat{a}_i\hat{a}_j - \hat{a}_j\hat{a}_i)\,|\,N_0,\cdots,N_i,\cdots N_j,\cdots\rangle$$
$$= \begin{cases} \sqrt{N_i(N_i-1)}|\,N_0,\cdots,N_i-2,\cdots\rangle - \sqrt{N_i(N_i-1)}|\,N_0,\cdots,N_i-2,\cdots\rangle = 0, & \textit{if } i=j, \\ \sqrt{N_iN_j}|\cdots,N_i-1,\cdots N_j-1,\cdots\rangle - \sqrt{N_iN_j}|\cdots,N_i-1,\cdots N_j-1,\cdots\rangle = 0, & \textit{if } i\neq j. \end{cases} \tag{1}$$

Since this is true for all test Fock states, we have shown $\hat{a}_i\hat{a}_j - \hat{a}_j\hat{a}_i = 0$ as an operator. For $\left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right]$ the proof is very similar. Finally

$$\left(\hat{a}_i\hat{a}_j^\dagger - \hat{a}_j^\dagger\hat{a}_i\right)|\,N_0,\cdots,N_i,\cdots N_j,\cdots\rangle$$
$$= \begin{cases} \sqrt{(N_i+1)(N_i+1)}|\,N_0,\cdots,N_i,\cdots\rangle - \sqrt{N_iN_i}|\,N_0,\cdots,N_i,\cdots\rangle = |\,N_0,\cdots,N_i,\cdots\rangle, & \textit{if } i=j, \\ \sqrt{N_i(N_j+1)}|\cdots,N_i-1,\cdots N_j+1,\cdots\rangle - \sqrt{N_i(N_j+1)}|\cdots,N_i-1,\cdots N_j+1,\cdots\rangle = 0, & \textit{if } i\neq j. \end{cases} \tag{2}$$

Since this is true for all test Fock states, we have shown $\hat{a}_i\hat{a}_j - \hat{a}_j\hat{a}_i = \delta_{ij}$ as an operator.

(ii) Show that the anti-symmetry of the Fermionic two-mode state $\langle x \,|\, 11 \rangle$ (see second dotpoint below Eq. (2.7)) under exchange of the two mode-labels a and b is correctly captured when using definition (2.6) and incorrectly when skipping the factor $(-1)^{\sum_{k<n} N_k}$.

Solution: The position space representation of this state is $\langle \mathbf{x} \,|\, 11 \rangle = \frac{1}{2}\big(\phi_a(\mathbf{x}_1)\phi_b(\mathbf{x}_2) - \phi_b(\mathbf{x}_1)\phi_a(\mathbf{x}_2)\big)$. *This is anti-symmetric under exchange of the two-particles* $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$*, which makes it automatically also anti-symmetric under exchange of the two state labels* $a \leftrightarrow b$.

Now consider Fock-states $|\, n_a, n_b \,\rangle$. *We can build* $|\, 1, 1 \,\rangle$ *as*

$$|\, 1, 1 \,\rangle = \hat{a}_a^\dagger \hat{a}_b^\dagger |\, 0, 0 \,\rangle \tag{3}$$

from the vacuum, since $\hat{a}_b^\dagger |\, 0 \,\rangle \overset{Eq.\ (2.5)}{=} (-1)^0 |\, 0, 1 \,\rangle$ *and then* $\hat{a}_a^\dagger |\, 0, 1 \,\rangle \overset{Eq.\ (2.5)}{=} (-1)^0 |\, 1, 1 \,\rangle$. *Now if we do* $a \leftrightarrow b$ *on the rhs of Eq. (3), we get*

$$\hat{a}_b^\dagger \hat{a}_a^\dagger |\, 0, 0 \,\rangle \overset{Eq.\ (2.5)}{=} \hat{a}_b^\dagger (-1)^0 |\, 1, 0 \,\rangle \overset{Eq.\ (2.5)}{=} (-1)^1 |\, 1, 1 \,\rangle = -|\, 1, 1 \,\rangle. \tag{4}$$

We could have also directly used the anti-commutator $\{\hat{a}_a, \hat{a}_b\} = 0$ *to see this, since it by design incorporates this behavior.*

(iii) Consider the Hamiltonian in second quantisation:

$$\hat{H} = \sum_m E_m \hat{a}_m^\dagger \hat{a}_m \tag{5}$$

where $\hat{a}_m$ destroy spin-1 bosons in a single (irrelevant) spatial mode and spin states $|\, s = 1, m_s = m \,\rangle$, $m = -1, 0, 1$ i.e. we are using eigenstates of $\hat{S}_z$ as single particle basis. What is the physical meaning of this Hamiltonian?

Now convert this Hamiltonian into one based on the single particle basis of eigenstates of $\hat{S}_x$, calling the corresponding operators $\hat{b}_m$, where $\hbar m$ is the eigenvalue of $\hat{S}_x$.

Solution: The physical meaning of the original Hamiltonian is just that each boson in state m *has an energy* E_m *that depends on its spin-state, e.g. due to an external magnetic field and the Zeeman effect.*

The spin operator $\hat{S}_x$ *for spin-1 has the matrix form*

$$\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6}$$

with eigenvectors $[1, \sqrt{2}, 1]^T/2$ $(m = +1)$, $[1, 0, -1]^T/\sqrt{2}$ $(m = 0)$, $[1, -\sqrt{2}, 1]^T/2$ $(m = -1)$. *The matrix containing these three eigenvectors as columns* $u_{\ell m}$ *is the transformation matrix between the two bases.*

Using Eq. (2.20) [with $\ell \leftrightarrow m$*]*

$$\hat{H} = \sum_{m\ell\ell'} E_m (u_{m\ell}^* \hat{c}_\ell^\dagger)(u_{m\ell'} \hat{c}_\ell') = \sum_{\ell\ell'} h_{\ell\ell'} \hat{c}_\ell^\dagger \hat{c}_\ell \tag{7}$$

where we have defined new single particle matrix elements of the Hamiltonian $h_{\ell\ell'} = \sum_m E_m u^*_{m\ell} u_{m\ell'}$.
E.g. for $E_m \to \kappa[1, 0, -1]^T$, *as would be the structure of the Zeeman effect, we have* $\hat{H} = \frac{\kappa}{\sqrt{2}}(\hat{c}^\dagger_1 \hat{c}_0 + \hat{c}^\dagger_0 \hat{c}_{-1} + h.c.)$. *We can see that the spin of all particles starting in an eigenstate of* $\hat{S}_x$ *will precess around the magnetic field direction, as expected.*

Stage 3 Solution comes later