PHY635, II-Semester 2022/23, Assignment 2

Instructor: Sebastian Wüster
Due-date: Email to TA by 27.1.2023

(1) Continuity equation for field operators: Consider a gas of interacting Bosons described by the Hamiltonian

$$\hat{H} = \int dx\ \hat{\Psi}^\dagger(x) \underbrace{\left[-\frac{\hbar^2}{2m}\nabla_x^2 + V(x)\right]}_{\equiv \hat{H}_o} \hat{\Psi}(x) + \frac{1}{2}\int dx \int dy\ \hat{\Psi}^\dagger(y)\hat{\Psi}^\dagger(x)U(x-y)\hat{\Psi}(x)\hat{\Psi}(y) \tag{1}$$

in field operator notation.

(i) Give a detailed derivation of the Heisenberg equation for the field operator $\hat{\Psi}(x)$, i.e. Eq. (2.33). [2pts]

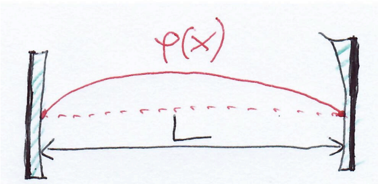
(ii) Define a scalar density operator $\hat{n}$ and a vector current operator $\hat{\mathbf{j}}$ such that the continuity equation

$$\frac{\partial}{\partial t}\hat{n} + \boldsymbol{\nabla}\cdot\hat{\mathbf{j}} = 0. \tag{2}$$

is fulfilled (and show that is is fulfilled). [5 pts]

(iii) Discuss what information is contained in Eq. (2) that goes beyond the continuity equation in classical field theory (e.g. fluid dynamics) [3 pts].

(2) Laser light: Consider a single mode photon field $\hat{\Psi}(x) = \varphi(x)\hat{a}$ within a laser cavity as shown in the diagram below.



Let the state describing the number of photons in the cavity be a coherent many-body state as in Eq. (2.50) of the lecture:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{3}$$

(i) What is the mean photon number in this state and what is its uncertainty? What is the probability distribution of photon number? [2 pts]

(ii) Let the Hamiltonian for this system be $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$. Find the equation of motion for $\hat{a}$ in the Heisenberg picture. [2 pts]

(iii) Using the relation $\hat{\mathcal{E}}(x) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\hat{\Psi}(x) - \hat{\Psi}^\dagger(x)\right)$ to define an electric field operator[1] $\hat{\mathcal{E}}(x)$ in terms of $\hat{\Psi}(x)$ at $t = 0$ (Heisenberg picture), find an expression for the mean electric field in the cavity as a function of time $t > 0$ and space, assuming the quantum state is the coherent state above. Identify a quantity that you can call the "phase" of that electric field. [4 pts] *Note, the notation differs from that of Example C on page 26.*

(iv) What is the uncertainty of the electric field at time $t = 0$ in the coherent state? [4 pts]

(3) Numerical evaluation of Wigner function: [4 points]

(i) Let us present the Fock space for a single mode for a restricted maximum number of particles $N_{\max}$ through a vector in $\mathbb{I}^{N_{\max}+1}$. This means for $N_{\max} = 2$, $|\,0\,\rangle \rightarrow [1,0,0]^T$, $|\,1\,\rangle \rightarrow [0,1,0]^T$, $|\,2\,\rangle \rightarrow [0,0,1]^T$. Using this, write down matrix representations for the creation and destruction operators and a single mode many-body density matrix. [2pts]

(ii) Combine these two results, to adjust the template MATLAB[2] script `Assignment2_wignerfct_v2.m`, such that it can plot the Wigner function of an arbitrary state. Use this to plot the Wigner function of a time evolving coherent state as in Example 13 of the lecture at a couple of representative times. Discuss how the evolution seen makes sense in relation with the discussion in Q2(d). Now plot the Wigner function of a Fock-state $|\,n = 5\,\rangle$. Can we attribute a phase to the corresponding electric field in the laser cavity as in Q2(d)? [6pts]

[1] See e.g. Walls and Milburn quantum optics for (too much) further information.

[2] Feel free to translate the script into mathematica if you prefer.