PHY635, II-Semester 2022/23, Assignment 2 solution

Instructor: Sebastian Wüster

(1) Continuity equation for field operators: Consider a gas of interacting Bosons described by the Hamiltonian

$$\hat{H} = \int dx\ \hat{\Psi}^\dagger(x) \underbrace{\left[-\frac{\hbar^2}{2m}\nabla_x^2 + V(x)\right]}_{\equiv \hat{H}_o} \hat{\Psi}(x) + \frac{1}{2}\int dx \int dy\ \hat{\Psi}^\dagger(y)\hat{\Psi}^\dagger(x)U(x-y)\hat{\Psi}(x)\hat{\Psi}(y) \tag{1}$$

in field operator notation.

(i) Give a detailed derivation of the Heisenberg equation for the field operator $\hat{\Psi}(x)$, i.e. Eq. (2.33). [2pts]

Solution: The Heisenberg equation for any operator is $i\hbar\dot{\hat{O}} = [\hat{O}, \hat{H}]$, *choosing* $\hat{O} = \hat{\Psi}(x)$ *this becomes:*

$$\begin{aligned} i\hbar\dot{\hat{\Psi}}(x) &= [\hat{\Psi}(x), \hat{H}] \\ &= \Bigg[\hat{\Psi}(x), \int dx' \hat{\Psi}^\dagger(x') \underbrace{\left[-\frac{\hbar^2}{2m}\nabla_{x'}^2 + V(x')\right]}_{\equiv \hat{H}_o(x')} \hat{\Psi}(x') \\ &\quad + \frac{1}{2}\int dx' \int dy\ \hat{\Psi}^\dagger(y)\hat{\Psi}^\dagger(x')U(x'-y)\hat{\Psi}(x')\hat{\Psi}(y)\Bigg] \end{aligned} \tag{2}$$

Using the bi-linearity of the commutator, we can take it into the integrations. There the field operator commutes with itself and also with $U(x-y)$ $(\hat{H}_0)$*, since the latter act in a two-particle particle (single-particle) Hilbertspace, while the field operator acts on Fockspace. Hence the only possibly nonzero commutators are*

$$= \int dx' \underbrace{\left[\hat{\Psi}(x), \hat{\Psi}^\dagger(x')\right]}_{=\delta(x-x')} \hat{H}_o(x')\hat{\Psi}(x') + \frac{1}{2}\int dx' \int dy \left[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)\hat{\Psi}^\dagger(x')\right] U(x'-y)\hat{\Psi}(x')\hat{\Psi}(y) \tag{3}$$

Using

$$\begin{aligned} \left[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)\hat{\Psi}^\dagger(x')\right] &= \left[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)\right]\hat{\Psi}^\dagger(x') + \hat{\Psi}^\dagger(y)\left[\hat{\Psi}(x), \hat{\Psi}^\dagger(x')\right] \\ &= \delta(x-y)\hat{\Psi}^\dagger(x') + \delta(x-x')\hat{\Psi}^\dagger(y), \end{aligned} \tag{4}$$

we find

$$\begin{aligned} i\hbar\dot{\hat{\Psi}}(x) &= \hat{H}_o(x)\hat{\Psi}(x) \\ &\quad + \frac{1}{2}\int dx' \hat{\Psi}^\dagger(x')U(x'-x)\hat{\Psi}(x')\hat{\Psi}(x) + \frac{1}{2}\int dy \hat{\Psi}^\dagger(y)U(x-y)\hat{\Psi}(x)\hat{\Psi}(y) \end{aligned} \tag{5}$$

We can rename the dummy variable $x' \to y$ in the first integration, thus

$$i\hbar\dot{\hat{\Psi}}(x) = \hat{H}_o(x)\hat{\Psi}(x) + \int dy \hat{\Psi}^\dagger(y)U(y-x)\hat{\Psi}(y)\hat{\Psi}(x), \tag{6}$$

which is Eq. (2.33) of the lecture. (We also had to use $[\hat{\Psi}(x), \hat{\Psi}(y)]$ once more to swap two field operators in the second term of (5), as well as $U(x-y) = U(y-x)$).

(ii) Define a scalar density operator $\hat{n}$ and a vector current operator $\hat{\mathbf{j}}$ such that the continuity equation

$$\frac{\partial}{\partial t}\hat{n} + \boldsymbol{\nabla} \cdot \hat{\mathbf{j}} = 0. \tag{7}$$

is fulfilled (and show that is is fulfilled). [5 pts]
Solution: We can use $\hat{n} = \hat{\Psi}^\dagger(x)\hat{\Psi}(x)$ and $\hat{\mathbf{j}} = -\frac{i\hbar}{2m}(\hat{\Psi}^\dagger(x)\boldsymbol{\nabla}\hat{\Psi}(x) - h.c.)$ in analogy to the definition of the probability density and current in (QM-1, section 1.6.4.).
Then

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{n} &= \left(\frac{\partial}{\partial t}\hat{\Psi}^\dagger(x)\right)\hat{\Psi}(x) + \hat{\Psi}^\dagger(x)\frac{\partial}{\partial t}\hat{\Psi}(x) \\
&\overset{Eq.~(6)}{=} +\frac{i}{\hbar}\left(\hat{H}_o^\dagger(x)\hat{\Psi}^\dagger(x) + \int dy \hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(y)U(y-x)\hat{\Psi}(y)\right)\hat{\Psi}(x) \\
&\quad - \frac{i}{\hbar}\hat{\Psi}^\dagger(x)\left(\hat{H}_o(x)\hat{\Psi}(x) + \int dy \hat{\Psi}^\dagger(y)U(y-x)\hat{\Psi}(y)\hat{\Psi}(x)\right).
\end{aligned} \tag{8}$$

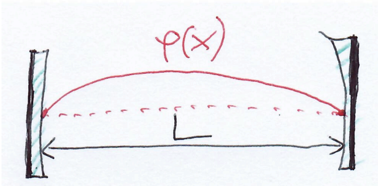
Expanding $\hat{H}_o(x) = -\hbar^2\boldsymbol{\nabla}^2/2m + V(x)$ and grouping terms:

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{n} &= \frac{i}{\hbar}\bigg[\left(-\frac{\hbar^2\boldsymbol{\nabla}}{2m}\hat{\Psi}^\dagger(x)\right)\hat{\Psi}(x) - \hat{\Psi}^\dagger(x)\left(-\frac{\hbar^2\boldsymbol{\nabla}}{2m}\hat{\Psi}(x)\right) \\
&\quad + \big[V(x) - V(x)\big]\hat{\Psi}^\dagger(x)\hat{\Psi}(x) + (1-1)\int dy \hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(y)U(y-x)\hat{\Psi}(y)\hat{\Psi}(x)\bigg] \\
&= -\boldsymbol{\nabla}\cdot\left[-\frac{i\hbar}{2m}(\hat{\Psi}^\dagger(x)\boldsymbol{\nabla}\hat{\Psi}(x) - \boldsymbol{\nabla}\hat{\Psi}^\dagger(x)\hat{\Psi}(x))\right] = -\boldsymbol{\nabla}\cdot\hat{\mathbf{j}},
\end{aligned} \tag{9}$$

which proves Eq. (7).

(iii) Discuss what information is contained in Eq. (7) that goes beyond the continuity equation in classical field theory (e.g. fluid dynamics) [3 pts].
Solution: The classical (mean field) version of Eq. (7) relates the (mean) density with the (mean) current. Here, we will additionally have relations between density fluctuations and current fluctuations.

(2) Laser light: Consider a single mode photon field $\hat{\Psi}(x) = \varphi(x)\hat{a}$ within a laser cavity as shown in the diagram below.



Let the state describing the number of photons in the cavity be a coherent many-body state as in Eq. (2.50) of the lecture:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{10}$$

(i) What is the mean photon number in this state and what is its uncertainty? What is the probability distribution of photon number? [2 pts]
Solution: The photon number operator is $\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k$ in general, but since we have only a single relevant mode this becomes just $\hat{N} = \hat{a}^\dagger \hat{a}$. We evaluate:

$$\langle \hat{N} \rangle = \underbrace{\langle\, \alpha\, | \hat{a}^\dagger}_{=\langle\, \alpha\, | \alpha^*} \; \underbrace{\hat{a} |\, \alpha\, \rangle}_{\overset{Eq.\ 2.43}{=} \alpha |\, \alpha\, \rangle} = |\alpha|^2 \langle\, \alpha\, |\, \alpha\, \rangle = |\alpha|^2. \tag{11}$$

We find the uncertainty as squareroot of the variance as usual, the latter is:

$$\langle \hat{N}\hat{N} \rangle - \langle \hat{N} \rangle^2 = \langle\, \alpha\, | \hat{a}^\dagger \underbrace{\hat{a}\hat{a}^\dagger}_{=\hat{a}^\dagger\hat{a}+1} \hat{a} |\, \alpha\, \rangle - (|\alpha|^2)^2 = |\alpha|^4 - |\alpha|^2 - |\alpha|^4 = |\alpha|^2. \tag{12}$$

The uncertainty is thus $\Delta N = \sqrt{|\alpha|^2} = \sqrt{\bar{N}}$, the squareroot of the mean number.
The probability distribution p_N of photon number N is given by $p_N = |c_N|^2$, where c_N are the coefficients in Eq. (10), hence

$$p_N = e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!}. \tag{13}$$

This is called Poisson distribution , which characteristically has a variance equal to the mean. (Not pronounced "poison" but "puassohn" after its French inventor.)

(ii) Let the Hamiltonian for this system be $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$. Find the equation of motion for $\hat{a}$ in the Heisenberg picture. [2 pts]
Solution: $i\hbar\dot{a}(t) = [\hat{a}, \hat{H}] = \hbar\omega\hat{a}$, which has the simple solution: $\hat{a}(t) = \hat{a}(0)e^{-i\omega t}$.

(iii) Using the relation $\hat{\mathcal{E}}(x) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\hat{\Psi}(x) - \hat{\Psi}^\dagger(x)\right)$ to define an electric field operator[1] $\hat{\mathcal{E}}(x)$ in terms of $\hat{\Psi}(x)$ at $t = 0$ (Heisenberg picture), find an expression for the mean

[1] See e.g. Walls and Milburn quantum optics for (too much) further information.

electric field in the cavity as a function of time $t > 0$ and space, assuming the quantum state is the coherent state above. Identify a quantity that you can call the "phase" of that electric field. [4 pts] *Note, the notation differs from that of Example C on page 26.*
Solution: More generally we would have $\hat{\Psi}(x) = \sum_n \varphi_n(x)\hat{a}_n$*, where* $\varphi_n(x)$ *are all the modes of the field, but since we were told to use a single-mode cavity model, this reduces to* $\hat{\Psi}(x) = \varphi(x)\hat{a}$*. From part (ii) we also know this as a function of time* $\hat{\Psi}(x,t) = \varphi(x)\hat{a}e^{-i\omega t}$*. Thus*

$$\begin{aligned}\langle\hat{\mathcal{E}}(x,t)\rangle &= \langle\,\alpha\,|i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\varphi(x)\hat{a}e^{-i\omega t} - \varphi^*(x)\hat{a}^\dagger e^{i\omega t}\right)|\,\alpha\,\rangle \\ &= i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\varphi(x)\alpha e^{-i\omega t} - \varphi^*(x)\alpha^* e^{i\omega t}\right). \qquad (14)\end{aligned}$$

Let us write $\alpha = |\alpha|e^{-i\phi}$ *for some phase* ϕ *and for simplicity assume real* $\varphi(x)$*, then*

$$\langle\hat{\mathcal{E}}(x,t)\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\varphi(x)|\alpha|e^{-i(\omega t+\phi)} - \varphi(x)|\alpha|e^{i(\omega t+\phi)}\right) = \sqrt{\frac{2\hbar\omega}{\epsilon_0}}|\alpha|\varphi(x)\sin\left(\omega t+\phi\right), \qquad (15)$$

where we see that the complex phase ϕ *of the coherent state provides the phase-offset of the oscillation of the electric field.*

(iv) What is the uncertainty of the electric field at time $t = 0$ in the coherent state? [4 pts]
Solution: We again use the variance

$$\begin{aligned}\langle\hat{\mathcal{E}}(x)\hat{\mathcal{E}}(x)\rangle - \langle\hat{\mathcal{E}}(x)\rangle^2 &= -\frac{\hbar\omega}{2\epsilon_0}\varphi(x)^2\langle\,\alpha\,|\left(\hat{a}-\hat{a}^\dagger\right)\left(\hat{a}-\hat{a}^\dagger\right)|\,\alpha\,\rangle \qquad (16) \\ &\quad - \frac{2\hbar\omega}{\epsilon_0}|\alpha|^2\varphi(x)^2\sin^2(\phi) \\ &= -\frac{\hbar\omega}{2\epsilon_0}\varphi(x)^2\left(\alpha^2 + \alpha^{*2} - |\alpha|^2 - \underbrace{\hat{a}\hat{a}^\dagger}_{=\hat{a}^\dagger\hat{a}+1}\right) - \frac{2\hbar\omega}{\epsilon_0}|\alpha|^2\varphi(x)^2\sin^2(\phi) \\ &= -\frac{\hbar\omega}{2\epsilon_0}\varphi(x)^2\left(|\alpha|^2e^{2i\phi} + |\alpha|^2e^{-2i\phi} - 2|\alpha|^2 - 1\right) - \frac{2\hbar\omega}{\epsilon_0}|\alpha|^2\varphi(x)^2\sin^2(\phi) \\ &= -\frac{\hbar\omega}{\epsilon_0}\varphi(x)^2\left(|\alpha|^2\left[\underbrace{\cos(2\phi)}_{1-2\sin^2(\phi)} - 1 - 2\sin^2(\phi)\right] - \frac{1}{2}\right) = \frac{\hbar\omega}{2\epsilon_0}\varphi(x)^2|\alpha|^2.\end{aligned}$$

and thus an uncertainty $\Delta\mathcal{E} = \sqrt{\frac{\hbar\omega}{2\epsilon_0}}|\varphi(x)||\alpha|$*. To check with literature*

(3) Numerical evaluation of Wigner function: [4 points]

(i) Let us present the Fock space for a single mode for a restricted maximum number of particles $N_{\max}$ through a vector in $\mathbb{I}^{N_{\max}+1}$. This means for $N_{\max} = 2$, $|\,0\,\rangle \rightarrow [1,0,0]^T$, $|\,1\,\rangle \rightarrow [0,1,0]^T$, $|\,2\,\rangle \rightarrow [0,0,1]^T$. Using this, write down matrix representations for the creation and destruction operators and a single mode many-body density matrix. [2pts]
Solution: The matrices are

$$\underline{\underline{\rho}} = \begin{pmatrix} \rho_{00} & \rho_{10} & \rho_{20} & \cdots \\ \rho_{01} & \rho_{11} & \rho_{21} & \cdots \\ \rho_{02} & \rho_{12} & \rho_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \underline{\underline{a}} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}, \underline{\underline{a}}^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{17}$$

We had written the single mode density matrix as $\hat{\rho} = \sum_{nm} \rho_{nm} |\,n\,\rangle\langle\,m\,|$, *where* $|\,n\,\rangle$ *are Fock states of that single mode. You can check by taking all matrix elements with the* $\mathbb{I}^{N_{max}+1}$ *vectors above that they work out identically.*

(ii) Combine these two results, to adjust the template MATLAB[2] script `Assignment2_wignerfct_v2.m`, such that it can plot the Wigner function of an arbitrary state. Use this to plot the Wigner function of a time evolving coherent state as in Example 13 of the lecture at a couple of representative times. Discuss how the evolution seen makes sense in relation with the discussion in Q2(d). Now plot the Wigner function of a Fock-state $|\,n=5\,\rangle$. Can we attribute a phase to the corresponding electric field in the laser cavity as in Q2(d)? [6pts]
Solution: You all did this well, so please refer to your own hand-in as solution. Anyone auditing pls. ask crediters, or google pertinent keywords.

[2]Feel free to translate the script into mathematica if you prefer.