4 DEGENERATE FERMI GASES

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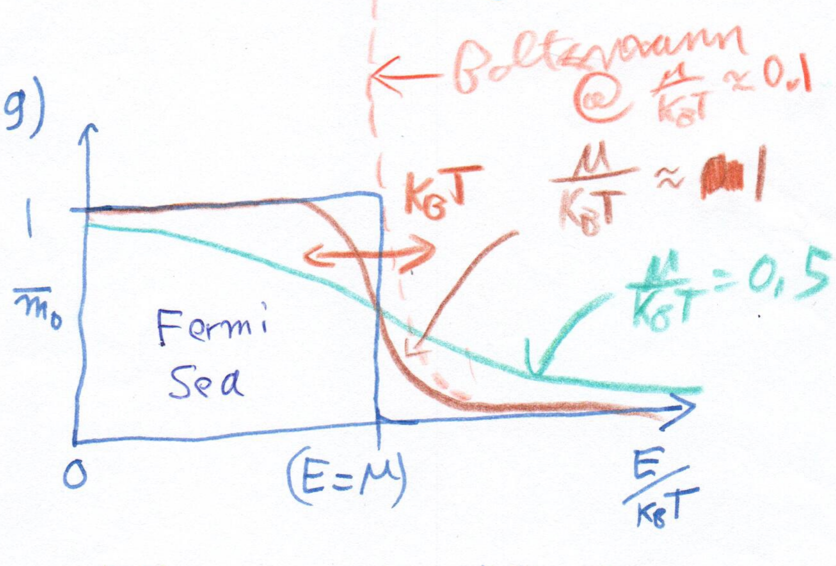
4.1. Ideal Fermi Gases

In section 3.2 we had asked what happens to N non-interacting Bosons as $T \downarrow 0$.
Now we follow the same question for Fermions, thus using the Fermi-Dirac distribution (3.9)

$$\overline{m}_b = \frac{1}{\exp(\beta(\varepsilon_b - \mu)) + 1}$$

- First difference: $\overline{m}_b > 0 \quad \forall\, \varepsilon_b, \mu \rightarrow$ no constraint on μ in contrast to Bose case.
- We can also take $\lim\limits_{T \to 0} \overline{m}_b = \begin{cases} +1 & \varepsilon_b < \mu \\ 0 & \varepsilon_b > \mu \end{cases} \qquad \left(\frac{1}{\exp(-\infty)+1}\right) \quad \left(\frac{1}{\exp(+\infty)+1}\right)$
- Plot of (3.9)

Fig1:

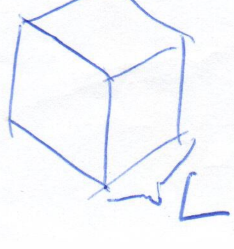


- Fermi-Dirac → Boltzmann as $k_B T \to \infty$

Let us consider $T=0$ case. As before μ sets the total number of particles (mean)

$$N = \sum_{\substack{\text{all states} \\ b}} \overline{m}_b$$

Let us look at non-interacting Fermions in a 3D box potential of cube-side-length with spins ($|S| = \frac{1}{2}$)



$$\Rightarrow \quad N = \sum_{n_x, n_y, n_z} \overline{m}_{n_x, n_y, n_z, s}$$

$$k_e = \frac{n_e \pi}{L}$$

$e \in \{x, y, z\}$

$n_x, n_y, n_z = 1, 2, \ldots \infty$

We have $\bar{m}_{n_x,n_y,n_z,s} = \begin{cases} 1 & \frac{\vec{n}^2\pi^2\hbar^2}{2mL^2} < \mu \\ 0 & \frac{\vec{n}^2\pi^2\hbar^2}{2mL^2} \geq \mu \end{cases}$ (using particle in the box energy $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$\Rightarrow$

$$N = \sum_{n_x,n_y,n_z,s} \bar{m}_{n_x,n_y,n_z,s} \approx \overset{\overset{s=\frac{1}{2}\text{ or }-\frac{1}{2}}{\downarrow}}{\frac{2}{8}} \int d^3\vec{n}\; \bar{m}_{\vec{n}} \underset{\substack{\text{spherical}\\ \text{co-ord}}}{\overset{3D}{=}} \frac{(4\pi)}{4} \int_0^{n_{max}} dn\, n^2 = \frac{1}{3}\pi n_{max}^3$$

originally $n_x,n_y,n_z > 0$ in $\int d^3\vec{n}$ +/−

where $n_{max} = \sqrt{\frac{2mL^2\mu}{\pi^2\hbar^2}}$

All together we obtain $N(\mu)$ and can solve for μ. $(V = L^3)$

This is also called Fermi-energy (for non-int $s=\frac{1}{2}$ Fermions in a box)

$$\mu_0 = E_F = (3\cdot\pi^2)^{2/3} \frac{\hbar^2}{2m} \left(\frac{N}{V}\right)^{2/3} \tag{4.1}$$

{$(6\pi^2)^{2/3}$ single state $(s=0)$} (needed later)

- Thus at $T=0$ (or for $k_BT \ll E_F$), the Fermions occupy all energy states up to (approx up to) E_F

  See blue line Fig. 1 (brown line Fig. 1)

  This configuration is called degenerate Fermi gas (DFG)

- The transition to a DFG is less sharp than for a BEC, roughly ~~transition~~ degeneracy temperature to DFG

$$k_BT_F \approx E_F \tag{4.2}$$

- We also define Fermi-momentum/wavenumber

  $n \leftarrow m$

$$\frac{\hbar^2 k_F^2}{2m} = \frac{p_F^2}{2m} = E_F, \tag{4.3}$$

$$k_F = \left[(3\pi^2)\,\rho\right]^{1/3} \qquad \rho = \frac{N}{V} \text{ density}$$

- Some examples:

Electrons in conductor: Fe mass density $\rho \sim 7.8\,\frac{g}{cm^3}$ $\Rightarrow$ atom number density $\rho_{Fe} \approx 8.3\cdot 10^{28}/m^3$

2 conduction e^- per atom $\rho_{e^-} = 16.6\cdot 10^{28}/m^3$ &(4.3)(4.1) $T_F \approx 1.3\cdot 10^5\,K$

$\Rightarrow$ Conduction electrons are DGF at all reasonable temperatures

$E_F \approx 2-10\,eV$

Cold fermionic atoms | density $n \approx 10^{17}/m^3$ (Using harmonic trap is better than box → see section 4.5)

$$T_F \approx 5 \cdot nK$$

- how cold is cold enough for DGF strongly depends on system

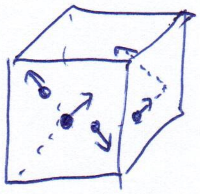
4.2. Degeneracy Pressure

One consequence of populating all states up to p_F, is that these particles may move "fast" and hence contribute to significant pressure.

Basic thermodynamic P, V, E relation

pressure!

$$P \cdot V = \frac{2}{3} N \langle \varepsilon_{kin} \rangle$$



pressure = elastic collisions off wall.

For the DGF: (Particles in box)

$$\langle \varepsilon_{kin} \rangle = \frac{\frac{2}{8}\int d^3\vec{n}\; E(\vec{n})}{N} \qquad \left(E(\vec{n}) = \frac{\vec{n}^2 \pi^2 \hbar^2}{2mL^2} \right)$$

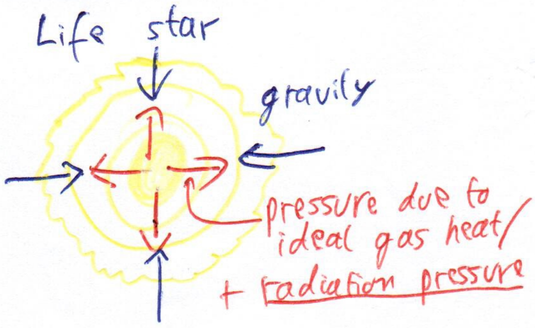
$$\underset{p.59}{\overset{see}{=}} \frac{(4\pi)}{4} \left(\frac{\pi^2 \hbar^2}{2mL^2} \right) \frac{\int_0^{n_{MAX}} dn\; n^4}{N} \xrightarrow{exercise} \frac{3}{5} E_F$$

We arrive at the Fermi-pressure (degeneracy pressure)

$$P_F = \frac{2}{5} \underbrace{\left(\frac{N}{V} \right)}_{= \rho} \cdot E_F \sim \rho^{5/3} \qquad (4.4)$$

- This is valid for $T \leq T_F$, and unlike the classical case, there is non-zero pressure all the way to $T = 0$
- You can think of this as Fermions resisting being squeezed into "same state". (But note, there are No Interactions)

4.3. Applications in Astrophysics



- In our sun, inward gravity is balanced by outward pressure and radiation pressure due to fusion reaction H + H → He sustaining temperature T.
- When fuel runs out, heavy stars shrink and get hotter, then burn He → C, ... Fe

- Won't work for solar-mass star, what happens when H is out?
  ↳ In some cases white dwarf with gravity balanced by Fermi pressure (4.4)

Stellar DGF:

Assume compressed star with $M = 10^{30}\,\mathrm{kg}$, $\overset{\text{center}}{\rho} = 10^{10}\,\mathrm{kg/m^3}$, $T = 10^7\,\mathrm{K}$

[c.f. sun $M_\odot = 2\cdot 10^{30}\,\mathrm{kg}$, $\rho_{\text{center}} = 1.6\cdot 10^5\,\mathrm{kg/m^3}$, $T = 1.57\cdot 10^7\,\mathrm{K}$]

Content: Ionized Helium $\Rightarrow$ $M \approx N_{\text{elec}}\, m_e + N_{\text{nucleons}}\, m_p$

$$= N_{\text{elec}}\,(\underset{\underset{0}{\ll}}{m_e} + 2\,m_p) = 2 N_{\text{elec}}\, m_p$$

turns out, pressure by He nucleii is negligible

Estimate number density of electrons roughly

$$\rho_e = \frac{N}{V} = \frac{M/2m_p}{M/\rho_{\text{center}}} \approx \frac{\rho_{\text{center}}}{2m_p} \approx 3\cdot 10^{-9}\,\frac{\text{electrons}}{\mathrm{fm}^3}$$

Fermi temperature $T_F \overset{(4.2)}{=} 8.8\cdot 10^9\,\mathrm{K}$ $\Rightarrow$

Despite being very hot, electrons at these high densities form DFG!

Stable equilibrium radius R of star:

$$0 = \frac{dE}{\cancel{dR}} = \frac{\partial}{\partial R}\Big(\underbrace{-\frac{3}{5}\frac{M^2}{R}G}_{\substack{E_{\text{grav}}\\ \text{(simplify star as}\\ \text{uniform sphere)}}}\Big)dR - \underbrace{P_F(\overset{\text{using } V(R)}{R})(4\pi R^2\, dR)}_{\substack{\text{using } dE = -PdV\\ \text{from thermodynamics}}} \qquad (4.5)$$

Can solve for white dwarf radius

$$\boxed{R_* = N\,\frac{\hbar^2}{G\, m_e\, m_{He}^{5/3}\, M^{1/3}} \qquad (4.6)}$$

- Proof exercise. Test: Sirius B, $M = 1.05\,M_\odot$, $R = 5100\,\mathrm{km}$
  (Formula (4.6) = 11000 km)

4.3.1. Relativistic DGF:

For very dense (massive) white dwarfs, e^- become relativistic. We have to re-calculate sections 4.1 / 4.2 using

$$E_{kin} = mc^2\left(\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right) \qquad (4.7)$$

Technical calculation, we find <u>relativistic Fermi-pressure</u>

$$P_F \sim \text{const} \cdot \rho^{4/3} \qquad (4.8)$$

If we redo (4.5) with this, we find there is <u>no stable R_*</u> for mass above <u>Chandrasekhar-limit</u>

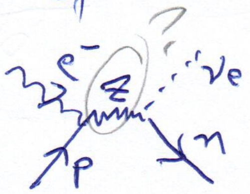
$$M \approx 1.44\, M_\odot \leftarrow \text{solar mass} \qquad (4.9)$$

(Maximal mass for white dwarf stars)

(Wiki $> 8 M_\odot$ initially, what's in between?)

4.3.2. Neutron stars:

- For heavier stars, electron degeneracy pressure cannot halt collapse.
- Once $\rho \sim 10^{17}\, kg/m^3$ (density of nuclei), electrons and protons form neutrons via



- Eventually <u>neutrons</u> become DGF and their P_F may halt collapse

  (At equal density P_F from neutrons is $\frac{m_e}{m_p}$ times that of electrons (much smaller), see Eq. (4.1))

- Result is called <u>neutron-star</u>, typically:

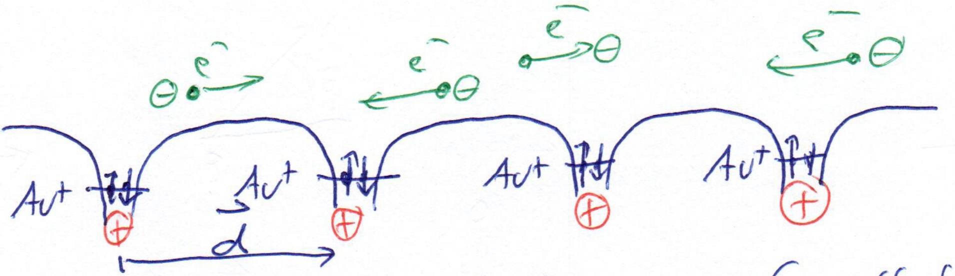
  $1.4 M_\odot < M < 3 M_\odot$, $R \sim 20\, km$

- If neutron Fermi-pressure is overcome as before (4.3.1) → total collapse, <u>black-hole</u>

  (If mass would be $> 10 M_\odot$ for sure)

4.4. Electron gas in metals

- Alkali-metal, Copper, Silver, Gold: 1 valence e^- per atom
- Picture:



- Ions bound by "immersion in electron gas (mettalic binding)
- electron-electron coulomb interactions are screened, due to background ion sea and hence weak
- electron-ion interactions: electrons aren't really "free", but see periodic potential $V(\vec{x}) = V(\vec{x}+\vec{d})$

Bloch theorem: $$\phi_{\vec{k},j}(\vec{x}) = u_j(\vec{x})\, e^{i\vec{k}\cdot\vec{x}} \quad (4.10)$$

where $u_j(\vec{x}) = u_j(\vec{x}+\vec{d})$, i.e. the function u possesses the same peiodicity properties as the potential $V(\vec{x})$

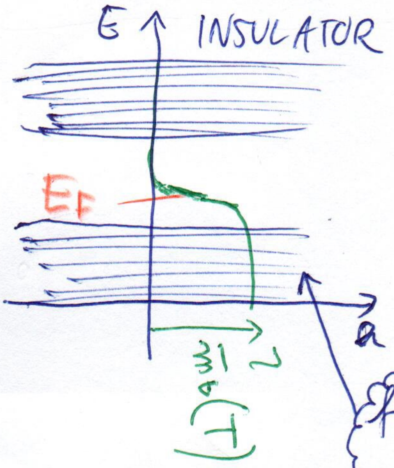
Gives rise to band-structure:

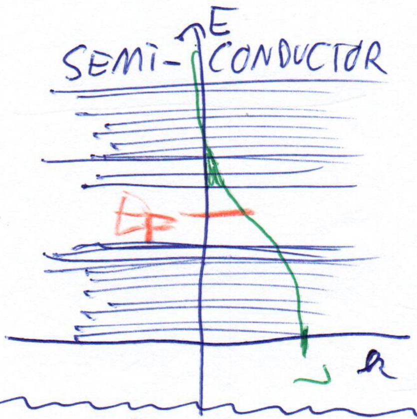
here they don't

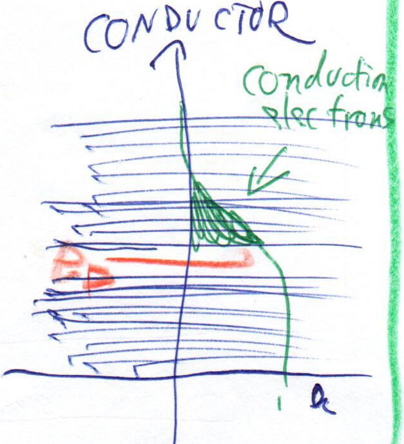
solutions like (4.10) of SE exist

- From numbers in example on p.59; valence electrons form a degenerate Fermi-gas

3 pictures distinguish:





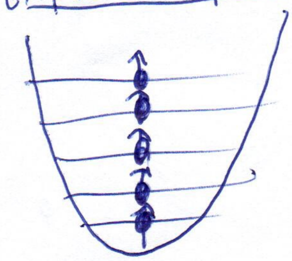


filled band doesn't conduct charge, cause for each e^- w. $\vec{k}$

4.5. Ultra-cold atomic Fermi-gas

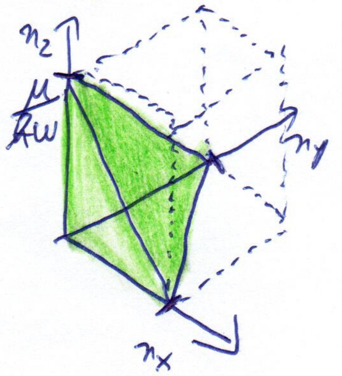
- As in 3.2 we now focus on a dilute gas of atoms in a harmonic trap, but now <u>fermionic atoms</u> ($N_{elec} + N_{nuclei} = odd$). $\Rightarrow N_{neutrons} = odd$
- We assume <u>spin-polarisation</u> (e.g. all ↑)
- We neglect interactions (but show shortly this is even realistic when all $|\uparrow\rangle$)

<u>Expected picture:</u>



- At $t = 0$ $\overline{n}_\epsilon = 1$ up to $\mu = E_F$
- Harmonic trap $E_{n_x, n_y, n_z} = \hbar\omega (n_x + n_y + n_z)$

$$\Rightarrow N(\mu) = \sum_{\substack{n_x, n_y, n_z \\ (\text{with } (n_x+n_y+n_z) < \frac{\mu}{\hbar\omega}}} 1 \approx \text{Volume of}$$



$$= \left(\frac{\mu}{\hbar\omega}\right)^3 / 6$$

$\Rightarrow$

Fermi-energy in trap!

$$\boxed{E_F = \hbar\omega\, (6N)^{\frac{1}{3}} \qquad (4.11)}$$

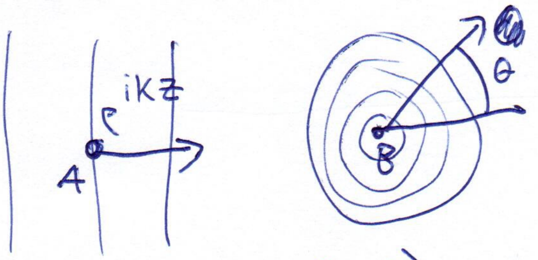
- Numbers as for Bose-atoms on p. 31! $T_F \approx 187\,nK$ ($N = 10\,000$, $\omega = (2\pi)\, 100\,Hz$)

- seems "easier" than BEC. Is in fact harder.
  <u>Reason:</u> (i) Evaporative cooling (see PHY 402 p. 89) relies on <u>interactions</u> for remnant atoms to rethermalize. (ii) Fermi-blocking (section 2.2.4): atom has to scatter exactly into the right "<u>empty</u>" state.
  We see in the next section that spin-polarized ultracold Fermions barely interact.

- Solution: e.g. sympathetic cooling: Mix Bosons & Fermions, cool Bosons, Fermions <u>can</u> interact with Bosons, thus cool ~~to~~ down together.

4.6. Ultra-cold Fermion interactions

Let us re-visit quantum scattering theory (as on page 32)



The wavefunction corresponding to this cartoon is

$$\psi_0(\vec{r}) = \exp(ikz) + \frac{f(\theta)}{r}\exp(ikr)$$

where $\vec{r} = \vec{r}_B - \vec{r}_A$ is the relative co-ordinate between the two collision partners, and $r, \theta\ (\varphi)$ the corresponding 3D spherical coordinates.

For <u>Fermions</u> need $\psi(\vec{r}) \overset{!}{=} -\psi(-\vec{r})$.

We could try the usual trick:

$$\psi(\vec{r}) = \frac{1}{\sqrt{2}}\left(\psi_0(\vec{r}) - \psi_0(-\vec{r})\right) \qquad (4.12)$$

Note $\psi_0(-\vec{r}) = \exp(-ikz) + \frac{f(\theta+\pi)}{r}\exp(ikr)$

But for s-wave scattering (see p. 32), $f(\theta) =$ const indep of θ so construction doesn't work. We would need $f(\theta) = -f(\theta+\pi)$, which would be true only for <u>p-wave scattering</u> ($l = 1$ relative angular momentum).

But our arguments to neglect p-wave scattering in the ultra-cold regime (p. 32) hold also for Fermions.

$\Rightarrow$ <u>Ultra-cold spin-polarized</u> fermions are to a very good approximation <u>non-interacting</u> (4.13)

- results as Eq. (4.11) are actually useful
- the situation changes if we have <u>2 spin-states</u> ↑, ↓, which can take care of <u>symmetrisation</u> in (4.12) $\Rightarrow$ then s-wave interactions <u>are</u> possible