7

3.4. Quasiparticles / quantized excitations

Section 3.3: We assumed the gas is <u>fully condensed</u> and effectively replaced $\hat{\Psi}(x) \Rightarrow \phi(x)$
operator — complex function

Let us now retain some possibly non-condensed atoms writing

<u>Field-operator with fluctuations</u>

$$\hat{\Psi}(x) = \phi_0(x) + \underbrace{\sum_n u_n(x)\hat{\alpha}_n - v_n^*(x)\hat{\alpha}_n^+}_{\hat{\chi}(x)\ \text{(c.f. Eq. 3.42)}} \qquad (3.44)$$

$\hat{\Psi}$ Bose atomic field operator

$\langle\hat{\Psi}\rangle = \phi_0$ (still) condensate mean field

$u_n(x), v_n(x)$ Bogoliubov mode function

$\hat{\alpha}_n, \hat{\alpha}_n^+$ Bogoliubov creation and destruction operators [bosonic]

$\hat{\chi}(x)$ fluctuation operator, <u>assumed small</u> ($\mathcal{O}(\hat{\chi}^3) = 0$)

We now insert (3.44) into Hamiltonian (3.25) and chose u_n, v_n such that the Hamiltonian is <u>diagonalized</u>.

<u>Diagonalized:</u> in terms of Fock states for Bogoliubov operators, this thus means $\hat{H} \sim \sum_n E_n \hat{\alpha}_n^+ \hat{\alpha}_n$

This is achieved when u_n, v_n fulfill the

<u>Bogoliubov-de-Gennes</u> equations (BdG) (keep on board)
(derivation: as for (3.43))

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x) + 2U_0|\phi_0(x)|^2 - \mu - \hbar\omega_n\right]u_n(x) - U_0\phi_0(x)^2 v_n(x) = 0$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x) + 2U_0|\phi_0(x)|^2 - \mu + \hbar\omega_n\right]v_n(x) - U_0\phi_0^{*2}(x) u_n(x) = 0 \qquad (3.45)$$

and <u>orthonormality conditions</u>

$$\int d^3x\, \phi_0^*(x) u_n(x) = \int d^3x\, \phi_0(x) v_n^*(x) = 0 \quad \text{(modes are orthogonal to condensate)}$$

$$\int d^3x \left[u_n(x) u_m^*(x) - v_n(x) v_m^*(x)\right] = \delta_{nm} \qquad (3.46)$$

$$\int d^3x \left[u_n(x) v_m(x) - v_n(x) u_m(x)\right] = 0$$

~~Notes~~

Using (3.45) the Hamiltonian takes the form

Quasi-particle Hamiltonian

$$\hat{H} = E[\phi] + \sum_n (\mu + \hbar\omega_n)\, \hat{\alpha}^+_n \hat{\alpha}_n \qquad (3.47)$$

$$E[\phi] = \int d^3x\, \phi^*(x)\left[-\frac{\hbar^2}{2m}\nabla + V(x) + U_0|\phi(x)|^2\phi(x)\right.$$

Gross-Pitaevskii energy functional

- Eq. (3.47) takes the form of a Hamiltonian for non-interacting entities created by $\hat{\alpha}^+_n$
- For that reason $\hat{\alpha}_n, \hat{\alpha}^+_n$ are called quasi-particle operators.
- Eq. (3.45) takes the same form as (3.43), which we got ~~from~~ starting with a seemingly quite different question. We will comment on this later.

- Using (3.46), we can "invert" (3.44) [exercise] to find

$$\hat{\alpha}_n = \int dx \left[u^*_m(x)\hat{\Psi}(x) + v^*_m(x)\hat{\Psi}^+(x)\right] \qquad (3.47b)$$

$$\hat{\alpha}^+_n = \int dx \left[u_m(x)\hat{\Psi}^+(x) + v_m(x)\hat{\Psi}(x)\right]$$

Hence we also call $u_m(x)$ particle amplitude

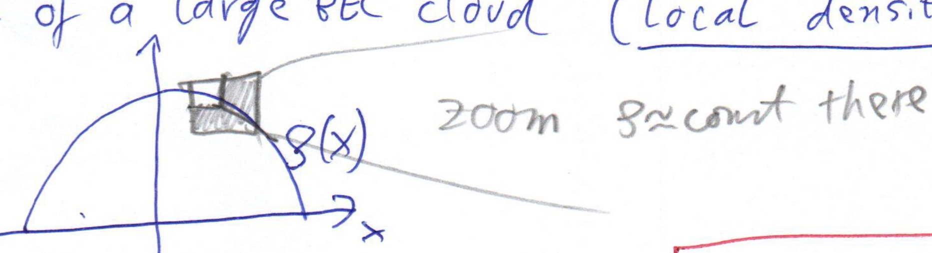
$v_m(x)$ hole amplitude

→ A BdG excitation is a superposition of added & substracted particles

3.4.1. Phonons

Let us proceed to solve the BdG equations (3.45) for the simple case of a <u>homogeneous, constant</u> condensate $\Rightarrow$ $\phi_0(x) = \sqrt{\rho}$ $\quad \rho =$ atom density
(indep of x.)

This can be realistic when concentrating on a small piece of a large BEC cloud (<u>Local density approximation</u> LDA)



For this case, we make the <u>plane-wave Ansatz</u>

discrete
($n \to q$)
continuous

$$\boxed{u_q(x) = \frac{u_q\, e^{iqx}}{\sqrt{V}} \qquad v_q(x) = \frac{v_q\, e^{iqx}}{\sqrt{V}}} \qquad (3.48)$$

- V is the quantisation volume
- q wave number
- u_q, v_q amplitude

Insert (3.48) into (3.45) and use $-\frac{\hbar^2}{2m}\nabla^2 u_q(x) = \underbrace{\frac{\hbar^2 q^2}{2m}}_{\equiv E_q} u_q$ etc.

We find matrix equation

$$\underbrace{\begin{bmatrix} E_q + 2U_0\rho - \mu - \hbar\omega_q & -U_0\rho \\ -U_0\rho & E_q + 2U_0\rho - \mu + \hbar\omega_q \end{bmatrix}}_{M} \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.49)$$

For non-trivial solution, need $\det(M) = 0$

$$\det(M) = -(\hbar\omega_q)^2 + (E_q + 2U_0\rho - \mu)^2 - U_0^2\rho^2 = 0$$

For a homogeneous condensate $\mu = U_0\rho$ (from Eq. (3.30) TIGPE)

$\Rightarrow$ We find for the excitations of the condensate the

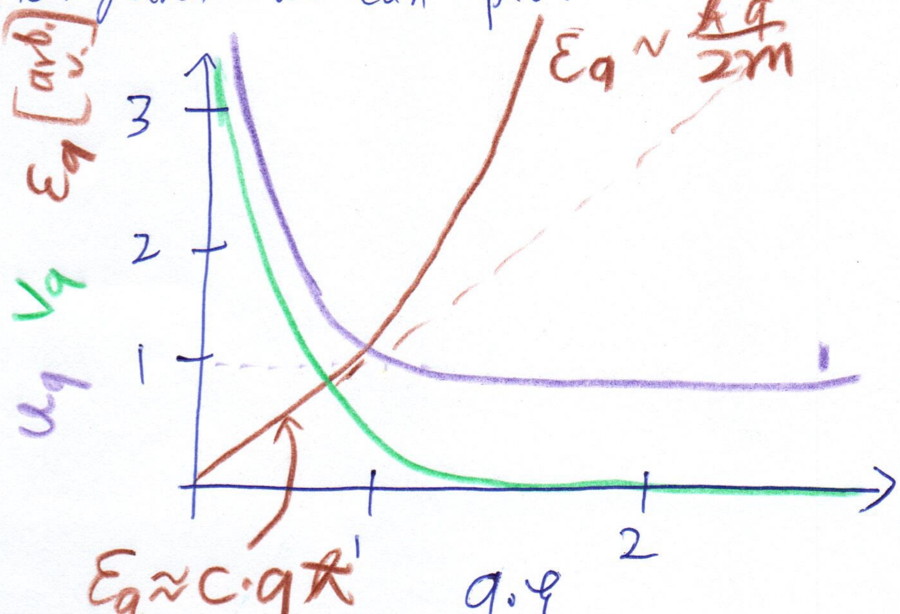
<u>Bogoliubov dispersion relation</u>

$$\boxed{\varepsilon_q \equiv (\hbar\omega_q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} + 2U_0\rho\right)} \qquad (3.50)}$$

Using (3.46), (3.49) and (3.50) we can show (using $\varphi_q \equiv E_q + U_0\rho$) that $u_q^2 = \frac{1}{2}\left(\frac{\varphi_q}{E_q}+1\right)$ $v_q^2 = \frac{1}{2}\left(\frac{\varphi_q}{E_q}-1\right)$ (3.51)

$u_q^2 - v_q^2 = 1$

Altogether we can plot



We have defined: $\boxed{\text{Speed of sound} \qquad c = \sqrt{\frac{U_0\rho}{m}} \quad (3.52)}$

Comments about Bogoliubov excitations:

- for $q \ll \xi$, we have $\mathcal{E}_q \approx c \cdot q\hbar$ and $|u_q|^2 + |v_q|^2 \gg 1$. $\mathcal{E}_q \approx c \cdot q\hbar$ is a linear dispersion relation as for sound-waves. $|u_q|^2 + |v_q|^2 \approx N_{atoms}$ involved in excitations (see *). So these are collective excitations / sound-waves for $q \ll \xi$
- for $q \gg \xi$, $\mathcal{E}_q \sim \frac{\hbar^2 q^2}{2m}$ like for a free particle. Also $|u_q|^2 + |v_q|^2 \to 1$. This is a single-atom excitation (~ 1 atom got kicked so hard, it no longer feels the others).

* Number of excited atoms $N_{exc} = \int \langle \hat{\chi}^\dagger \hat{\chi} \rangle dx$ (see (3.44))

Let $|\psi\rangle = |N_1 N_2 \dots\rangle$ be the Fock state for occupation of Bogoliubov excitations

$$\Rightarrow N_{exc} = \int_V dx \sum_{qq'} \Big(u_q^*(x)\, u_{q'}(x)\, \underbrace{\hat{\alpha}_q^\dagger \hat{\alpha}_{q'}}_{\delta_{qq'} \text{ in state } |\psi\rangle} + v_q(x) v_{q'}^*(x) \underbrace{\hat{\alpha}_q \hat{\alpha}_{q'}^\dagger}_{(\hat{\alpha}_{q'}^\dagger \hat{\alpha}_q + \delta_{qq'})} \Big) = \sum_q \left(|u_q|^2 + |v_q|^2\right) N_q + \sum_q |v_q|^2$$

3.4.2 Time-dependence

The overall time-dependence of the field operator in Eq. (3.44) is

Time-dependence of BdG modes

$$\boxed{\hat{\Psi}(x,t) = e^{-i\frac{\mu}{\hbar}t}\left[\phi_0(x) + \sum_n u_n(x)\hat{\alpha}_n e^{-i\omega_n t} - v_n^*(x)\hat{\alpha}_n^\dagger e^{i\omega_n t}\right] \quad (3.53)}$$

- c.f. Eq. (3.42)
- to see this insert (3.44) into $\overset{\text{Heisenberg}}{(3.26)}$ using $\overset{\text{GPE}}{(3.29)}$ and $\overset{\text{BdG}}{(3.45)}$

3.4.3 Coherent vs. incoherent excitation

Ⓐ In section 3.3.7: Slightly perturb GPE solution $\phi(x,t) = \phi_0(x) + \delta\phi(x,t)$, how does perturbation $\delta\phi$ evolve?

Ⓑ In section 3.4: In QFT problem, which fluctuation modes outside the BEC diagonalize the Hamiltonian?

Seemingly different questions give the same BdG equations for condensate excitations (3.43) vs (3.45)

The reason is, that Ⓐ is included in Ⓑ. Consider a single Bogoliubov mode only (say $n=1$). Assume its quantum state is $|\psi\rangle = |\beta\rangle$ $\beta \in \mathbb{C}$

$\uparrow$ coherent state.

Then

$$\langle\hat{\Psi}(x,t)\rangle \overset{\text{Eq. (3.53)}}{=} e^{-i\frac{\mu}{\hbar}t}\left[\phi_0(x) + u_1(x)\beta e^{-i\omega_1 t} - v_1^*(x)\beta^* e^{i\omega_1 t}\right]$$

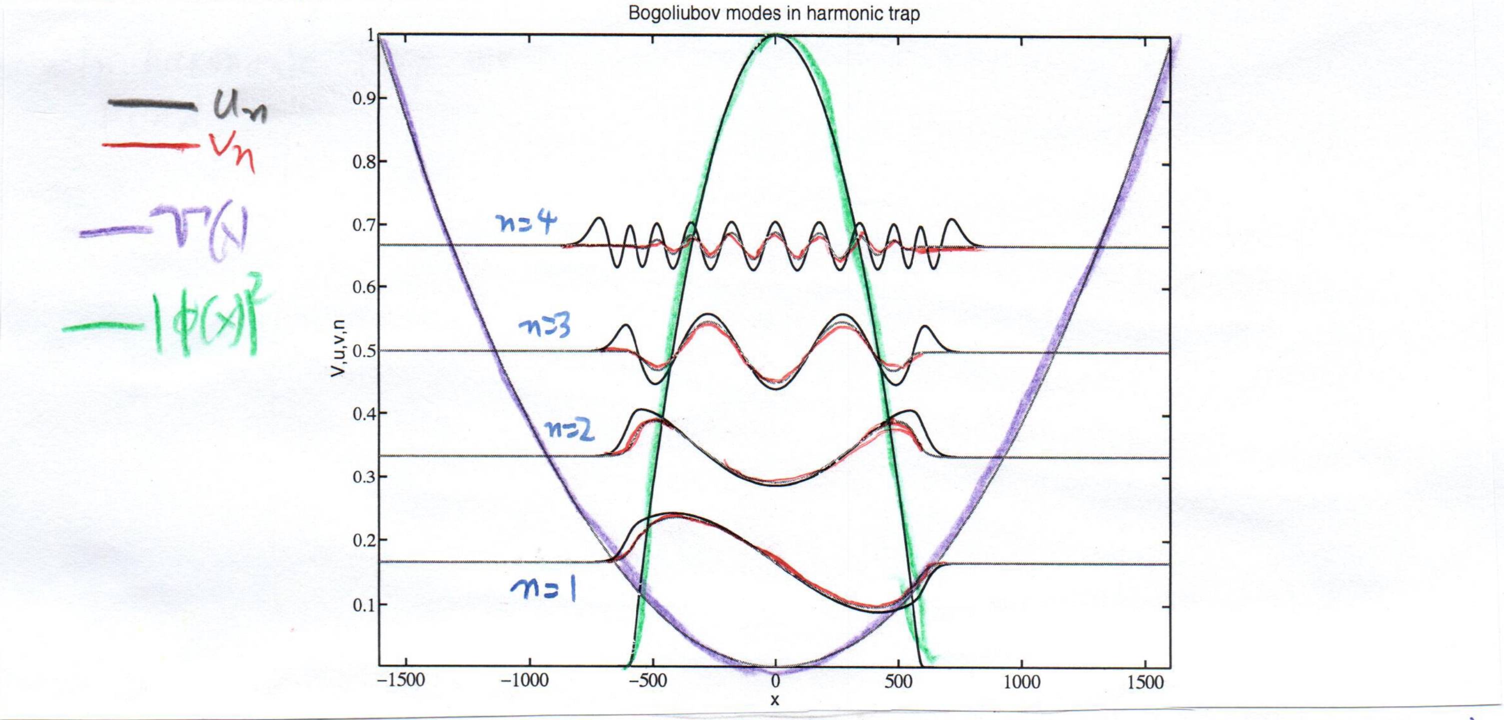
which is a BEC perturbation as in (3.42). (So here the population in mode 1 has phase-coherence with the BEC)

Had we used $\hat{\varrho} = \sum_n p_n |n\rangle\langle n|$ for mode 1, we keep $\langle\hat{\Psi}\rangle = e^{-i\frac{\mu}{\hbar}t}\phi_0(x)$, so this is incoherent thermal population.

$\updownarrow$
p_n

3.4.4. The thermal cloud

In general Eq. (3.45) has to be solved numerically (see Pethick & Smith for analytical approximation techniques).



As $n \mapsto \infty$, the modes approach $u_n \to \varphi_n$ (S.H.O states, (1.2))
$v_n \to 0$

(As for the homogeneous case, high energy BdG modes (phonons) become single-particle excitations.)

BEC experiments never reach $T=0$, hence we write

$$\boxed{\hat{\varrho} = \sum_{\vec{N}} P_{\vec{N}} \, |\vec{N}\rangle\langle\vec{N}| \qquad P_{\vec{N}} \text{ see } (3.5) \qquad (3.54)}$$

for the state of <u>thermal uncondensed</u> atoms

- We assume there is a (much larger) BEC component co-existing (Not described by (3.54), but (3.44) $\hat{\Psi} = \phi + \hat{\chi}$)

Total atom density

$$n(x) = \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \rangle = |\phi_0(x)|^2 + \overset{\mathrm{Tr}[\hat{\varrho}\hat{\alpha}]}{0} + \sum_{nn'} \mathrm{Tr}\Big[\hat{\varrho}\,\big(u_n^*(x)\hat{\alpha}_n^\dagger - v_n(x)\hat{\alpha}_n\big) \times \big(u_{n'}(x)\hat{\alpha}_{n'} - v_{n'}^*(x)\hat{\alpha}_{n'}^\dagger\big)\Big] \quad \nearrow \sim \delta_{nn'}$$

$$= |\phi_0(x)|^2 + \sum_n \mathrm{Tr}\Big[\hat{\varrho}\big\{\big(|u_n(x)|^2 + |v_n(x)|^2\big)\hat{\alpha}_n^\dagger\hat{\alpha}_n + |v_n(x)|^2\big\}\Big]$$

$\Rightarrow$ <u>Atom density</u>

$$\boxed{n_{tot}(x) = \underbrace{|\phi_0(x)|^2}_{\text{BEC}} + \sum_n \Big\{\underbrace{\big(|u_n(x)|^2 + |v_n(x)|^2\big)}_{\text{thermal cloud}} \underset{\substack{\downarrow \\ = \bar{m}_n \text{ (Eq. 3.10)}}}{n_{th}} + \underbrace{|v_n(x)|^2}_{\text{quantum fluctuations}}\Big\} \qquad (3.55)}$$

EXAMPLE:

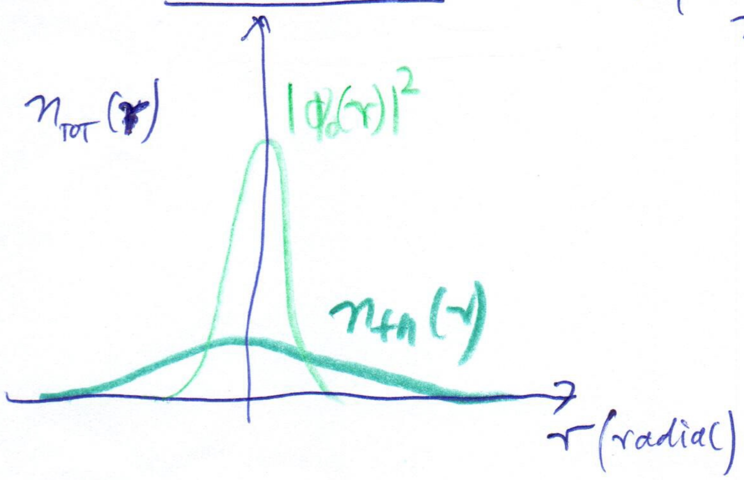
Approximate $V \approx 0$, $u_n \to \varphi_n$ ⟹ nasty calculation:

thermal cloud shape

$$n_{th}(x) = \frac{N_{th}}{\pi^{3/2} R_x R_y R_z} e^{-\frac{x^2}{2R_x^2}} e^{-\frac{y^2}{2R_y^2}} e^{-\frac{z^2}{2R_z^2}} \qquad (3.56)$$

$R_i = \sqrt{\frac{2 k_B T}{m \omega_i}}$, widths depend on temperature

Overall bi-modal density, can be used to measure T



3.4.5 Superfluidity

Phenomenological definition

(i) flow without friction through small capillaries

(ii) perfect heat conductivity (via convection)

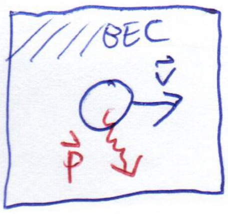
(iii) rotation only via quantized vortices (see section 3.3.6)

Found e.g. in dilute gas BEC & cold liquid helium.

Why?

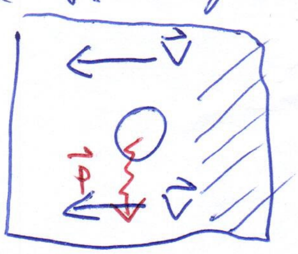
Critical velocity:

Consider a BEC through which we drag an obstacle (e.g. Laser potential $V(x,t)$) with velocity $\vec{v}$



Lab-frame

Friction would occur by dissipating energy towards exciting quasi-particles with momenta $\vec{p}$



Obstacle rest frame

Consider energy of ~~[illegible]~~ gas in the two frames (skip)

	Lab-frame	Rest frame
Ground-state (BEC only)	E' (some internal energy)	$E' + \frac{1}{2} N m v^2$ (N atoms, m mass of one atom)
Excited state (BEC plus one excitation $\vec{p}$)	$E' + \varepsilon_p$ (see Eq (3.50))	$E' + \varepsilon_p \underbrace{- \vec{p}\cdot\vec{v}}_{\text{Doppler shift}} + \frac{1}{2} N m v^2$
	$V(x,t)$, Hamiltonian time-dependent energy NOT conserved	$V(x)$, energy conserved, can only create excitation if $\Delta E > 0$

ΔE

Energy needed to create excitation:

$$\boxed{\Delta E = \varepsilon_p - \vec{p}\cdot\vec{v}} \quad (3.57)$$

Smallest gap at $\vec{p} \parallel \vec{v} \Rightarrow 0 = \varepsilon_p - |\vec{p}|\,|\vec{v}_{crit}|$ critical velocity for $\Delta E = 0$

$\Rightarrow V_{crit} = \mathrm{Min}_{\vec{p}} \left[\frac{\varepsilon_p}{|\vec{p}|} \right]$, below V_{crit} cannot create excitation

From Eq. (3.50):

Critical velocity (below V_{crit}, there is superfluidity)

$$V_{crit} = \mathrm{Min}_{\vec{p}} \left[\frac{\sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2U_0 \rho\right)}}{p} \right] = \sqrt{\frac{\rho U_0}{m}} = c \quad \text{speed of sound} \quad (3.58)$$

$(\vec{p} \Leftrightarrow \hbar\vec{q})$

- In a usual fluid, there a single particle excitations $\varepsilon_p \sim \frac{p^2}{2m}$ for arbitrarily small p (unlike here). $\Rightarrow$ No superfluidity
- Thus superfluidity relies on interactions

3.5. Condensate stability

Lets return to (3.42) (Mean-field perturbations). [same conclusions follow from (3.44)] $\phi(x,t) = e^{-i\mu t/\hbar}\left[\phi_0(x) + u(x)e^{-i\omega t} - v^*(x)e^{i\omega t}\right]$

- BdG eqn (3.43) does not guarantee $\omega \in \mathbb{R}$, can be $\omega \in \mathbb{C}$.

Example: Homogeneous condensate with attractive interactions $U_0 < 0$

$$\hbar\omega_q = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} + 2U_0\rho\right)}$$

$Im(\hbar\omega_q) \neq 0$ for $q < \frac{\sqrt{4|U_0|\rho m}}{\hbar}$

- they also don't guarantee $Re[\omega] > 0$ (which means excitation has in fact higher energy than BEC). Write $\omega = \underset{Re}{\omega'} + i\underset{Im}{\omega''}$

We can classify 3 cases:

$\omega' > 0,\ \omega'' = 0$ Usual stable case, modes oscillatory

$\omega'' \neq 0$ Condensate dynamically (modulationally) unstable

Small perturbation in Eq. (3.42) will grow exponentially with growth rate $\sim |\omega''|$

Examples: homogeneous $U_0 < 0$ $\rightarrow$ bright solitons
rotated BEC $\rightarrow$ vortices
BEC $U_0 > 0$ optical-lattice band-gap $\rightarrow$ gap-solitons

Usually the end-product of this instability is a new (stable) non-linear solution of TIGPE these

$\omega' < 0$ Condensate energetically unstable

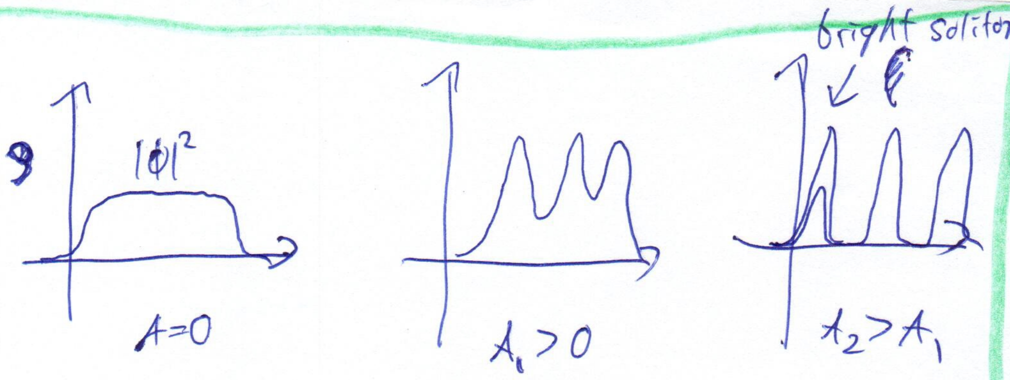
- All is fine in (3.42) [which assumes unitary evolution] but $\phi_0(x)$ is NOT a local minimum of $E[\phi]$ (3.47) Hence any dissipation will destroy $\phi_0(x)$

Examples:

(i) BEC collapse

$U_0 < 0$

dynamically and energetically unstable



Bright solitons: Non-linear solution of TIGPE for $U_0<0$, $V=0$

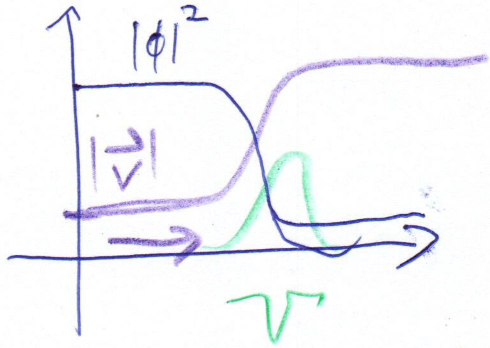
Using $\phi_0 \sim \mathrm{sech}(x)$ [soliton], all BdG modes from (3.43) are stable

(ii) Supersonic flow

$U_0 > 0$

$V_{flow} > c$

May be dynamically stable but is energetically unstable



↗ follows from (3.38)/(3.39)

Again, doppler shift as in 3.4.5

$\hbar\omega' = \hbar\omega - VK$

⇒ V_{crit} previous section = some phonons become energetically unstable

↳ We would reach similar conclusions looking at quantized, incoherent excitations.

How do ~~they~~ all (Mean-field BEC, thermal & quantum excitations) play together in a time-dependent manner?

↳ Next chapter.