3.3. Gross-Pitaevskii equation

- Previously, we considered <u>non-interacting</u> Bose gas
- BEC occurs also if (weak) interactions are present
- these can be treated very simply for dilute-gas BEC

3.3.1 Contact Interactions

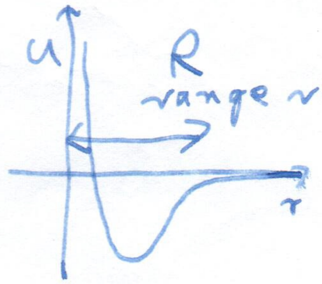
Consider N interacting Bosons with Hamiltonian (c.f. 2.23)

$$\hat{H} = \sum_{k=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla^2_{x_k} + V(x_k)\right) + \frac{1}{2}\sum_{k,l=1}^{N} U(\vec{x}_k - \vec{x}_l) \qquad (3.15)$$

with $V(x_k) = \frac{1}{2} m \omega^2 x_k^2$ (harmonically trapped).

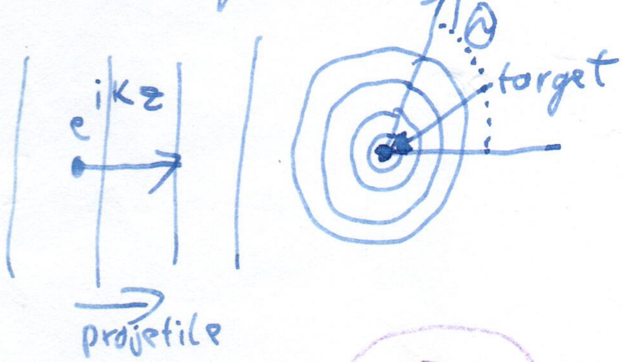
Realistically $U(\vec{x}_k - \vec{x}_l) = \frac{A}{r^{12}} - \frac{B}{r^6}$ $\quad r = |\vec{x}_k - \vec{x}_l|$ $\quad (3.16)$

(Lennard-Jones potential

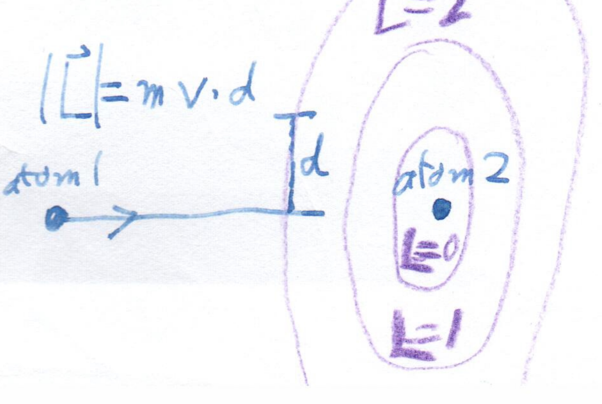


- Densities in BEC are such that mean distance $d \gg R$ $\Rightarrow$ detail shape of U irrelevant

Assuming two atoms do get close and collide, need quantum scattering theory.



See QM-book: We can expand incoming plane-wave in terms of angular momentum wrt. scattering target.



At low temperatures (low scattering velocities) only $\ell = 0$ (s-wave) will contribute

$\Rightarrow$

- This is the s-wave scattering approximation, with isotropic outgoing wave

Together the two yellow boxes on p. 32 allow the use of

$\boxed{\begin{array}{l}\text{contact interactions} \quad \text{(chosen to give same s-wave scattering as (3.16))} \\ U(\vec{x}_k - \vec{x}_\ell) \rightarrow U_0\, \delta(\vec{x}_k - \vec{x}_\ell) \qquad (3.17) \\ \text{with} \quad U_0 = \dfrac{4\pi\hbar^2 a_s}{m}\end{array}}$

- a_s is the s-wave scattering length, that quantifies the amplitude of the scattering process (or total cross-section)
- sign of a_s tells if interactions are $\begin{cases}\text{repulsive } a_s > 0 \\ \text{attractive } a_s < 0\end{cases}$

3.3.2. Condensate wave function

Let us assume sth. like (3.14) [$|\Psi_0\rangle = |N\,000\ldots\rangle$] holds even in the interacting case.

All Bosons in same state $\Rightarrow$

$\boxed{\begin{array}{l}\text{Ansatz for many-body wave-function (N Bosons, 1D)} \\ \Psi(\vec{x}, t) = \prod_{\ell=1}^{N} \phi(\vec{x}_\ell, t) \quad \text{with } \boxed{\int_{-\infty}^{\infty} dx\, |\phi(x,t)|^2 = 1} \qquad (3.18)\end{array}}$

Determine $\phi(x,t)$ from $\hat{H}$ (2.23) with $U = (3.17)$ using time-dependent variational principle: $\delta S \overset{\triangle}{=} 0$

$\boxed{\begin{array}{l}\text{Lagrangian density (gives Many-Body SE upon variation of } \Psi) \\ \text{Action} \\ S = \int dt\, d^N\vec{x} \left\{ \dfrac{i\hbar}{2}\left[\Psi^*(\vec{x},t)\dfrac{\partial}{\partial t}\Psi(\vec{x},t) - \Psi(\vec{x},t)\dfrac{\partial}{\partial t}\Psi^*(\vec{x},t)\right] - \Psi^*(\vec{x},t)\hat{H}\Psi(\vec{x},t) \right\} \qquad (3.19)\end{array}}$

- Variational principle allows us to "enforce" guess (3.18) and then ask "what equation does $\phi(x,t)$ have to follow?"

$$\underline{\text{Note:}}\ \frac{\partial}{\partial t}\Psi(\vec{x},t) \underset{\substack{(3.18) \\ \& \\ \text{Product rule}}}{=} \left[\sum_{k=1}^{N} \frac{\partial}{\partial t}\phi(x_k,t)\right] \prod_{\substack{\ell=1 \\ \ell\neq k}}^{N} \phi(x_\ell, t) \qquad (3.20)$$

Let us insert (3.18) into (3.19) and simplify

$$S \overset{(3.20)}{=} \int dt\, d^N x \Big\{ \frac{i\hbar}{2}\Big[\prod_{l'=1}^{N} \phi^*(x_{l'},t)\, \cancel{\prod_{l=1,\, l\neq k}^{N}} \sum_{k=1}^{N} \prod_{\substack{l=1\\ l\neq k}}^{N} \phi(x_l,t) \Big(\frac{\partial}{\partial t}\phi(x_k,t)\Big)$$

$$- \prod_{l'=1}^{N} \phi(x_{l'},t') \sum_{k=1}^{N} \prod_{\substack{l=1\\ l\neq k}}^{N} \phi^*(x_l,t) \Big(\frac{\partial}{\partial t}\phi^*(x_k,t)\Big)\Big] \qquad (3.21)$$

$$- \prod_{l'=1}^{N} \phi^*(x_{l'},t') \Big[\sum_{k=1}^{N} \hat{H}_0(x_k) + \frac{1}{2} \sum_{\substack{k,m=1\\ k\neq m}}^{N} U_0\, \delta(x_k - x_m) \Big] \prod_{l=1}^{N} \phi(x_l,t) \Big\}$$

We have to integrate $\int d^N x = \int dx_1 \int dx_2 \dots \int dx_N$. All terms with $x_i \neq x_k$ or x_m give $\int |\phi(x_l,t)|^2 dx_l = 1 \Rightarrow$

$$S = \sum_{k=1}^{N} \int dt \int dx_k \Big\{ \frac{i\hbar}{2}\Big[\phi^*(x_k,t)\frac{\partial}{\partial t}\phi(x_k,t) - \phi(x_k,t)\frac{\partial}{\partial t}\phi^*(x_k,t)\Big] \qquad (3.22)$$

$$- \phi^*(x_k,t)\, \hat{H}_0(x_k)\, \phi(x_k,t) + \underbrace{\frac{U_0}{2} \sum_{\substack{m=1\\ m\neq k}}^{N} \int dx_m\, \phi^*(x_k,t)\, \phi^*(x_m,t) \times \delta(x_k - x_m)\, \phi(x_k,t)\, \phi(x_m,t)}_{= \frac{U_0(N-1)}{2} \phi^*(x_k,t)^2 \phi(x_k,t)^2} \Big\}$$

Now, note all terms in $\sum_{k=1}^{N}$ are the <u>same</u>:

$$S = N \int dt \int dx \Big\{ \frac{i\hbar}{2}\, \phi^*(x,t)\frac{\partial}{\partial t}\phi(x,t) - \phi(x,t)\frac{\partial}{\partial t}\phi^*(x,t) \qquad (3.23)$$

$$- \phi^*(x,t)\hat{H}_0\phi(x,t) - \frac{U_0(N-1)}{2}\phi^*(x,t)^2\phi(x,t)^2 \Big\}$$

better: separately do for $\delta\phi^*$

Variation of L: $\delta L = L[\phi + \delta\phi, \phi^*] + L[\phi, \phi^* + \delta\phi^*] - 2L[\phi, \phi^*]$

(we treat ϕ, ϕ^* as independent). Demand

integration by parts

$$0 = \delta S = N \int dt \int dx\, \frac{i\hbar}{2}\Big\{ \delta\phi^* \frac{\partial}{\partial t}\phi + \phi^* \frac{\partial}{\partial t}\delta\phi - \delta\phi \frac{\partial}{\partial t}\phi^* - \phi \frac{\partial}{\partial t}\delta\phi^*$$

$$- \delta\phi^*\big[H_0\phi + U_0(N-1)\phi^*\phi^2\big] - \big[\phi^* H_0 \delta\phi + U_0(N-1)\phi^{*2}\phi\delta$$

Now this should be zero for all small <u>functions</u> $\delta\phi(x,t)$

$\Rightarrow$ Coefficient of $\delta\phi(x,t)$, $\delta\phi^*(x,t)$ must vanish inside integral $\forall x,t$

from coefficient of $\delta\phi^*(x,t)$:

Gross-Pitaevskii equation (GPE) (time-dependent)

$$i\hbar\,\dot{\phi}(x,t) = \Big(\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)}_{\hat{H}_0(x)} + U_0(N-1)\,|\phi(x,t)|^2\Big)\,\phi(x,t) \quad (3.24)$$

- $\phi(x,t)\in\mathbb{C}$ is the condensate wave-function (here $\int_{-\infty}^{\infty}|\phi(x,t)|^2 dx = 1$)
- $U_0 = \frac{4\pi\hbar^2 a_s}{m}$ from (3.17)
- Similar to single-particle Schrödinger equation but with nonlinear term
- $(N-1)\,|\phi(x,t)|^2 \sim \rho$ = atom density. Non-linear term = interactions

3.3.3 Mean-field theory

We can derive Eq. (3.24) differently, starting from quantum-field-theory

Consider $\hat{H}$ in (2.28) with $U(x-y)$ from (3.17)

$$\hat{H} = \int dx \left\{\hat{\Psi}^\dagger(x)\, H_0\, \hat{\Psi}(x) + \frac{U_0}{2}\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x)\,\hat{\Psi}(x)\hat{\Psi}(x)\right\} \quad (3.25)$$

Consider field operator in Heisenberg picture $\hat{\Psi}(x,t)$

Heisenberg equation for field operator

$$i\hbar\,\dot{\hat{\Psi}}(x,t) = \left[\hat{H},\hat{\Psi}(x,t)\right] = \hat{H}_0\,\hat{\Psi}(x,t) + U_0\,\hat{\Psi}^\dagger(x,t)\,\hat{\Psi}(x,t)\,\hat{\Psi}(x,t) \quad (3.26)$$

We ought to specify a (initial) quantum state, but let's not, rather we ~~[illegible]~~ make the:

Mean-field Ansatz:

$$\left\langle \hat{\Psi}(x,t)\right\rangle = \tilde{\phi}(x,t) \quad (3.27)$$

- $\tilde{\phi}$ is the condensate wavefunction (here $\int_{-\infty}^{\infty}|\tilde{\phi}(x,t)|\,dx = N$) or order-parameter of the BEC

Taking the expectation value of (3.26) we reach

$$i\hbar\dot{\phi}(x,t) = \hat{H}_0 \phi(x,t) + U_0 \langle \hat{\Psi}^\dagger(x,t)\hat{\Psi}(x,t)\hat{\Psi}(x,t)\rangle \qquad (3.28)$$

We now assume factorisation: $\overset{\shortparallel}{\langle\hat{\Psi}^\dagger(x,t)\rangle\langle\hat{\Psi}(x,t)\rangle\langle\hat{\Psi}(x,t)\rangle}$

and reach

$$\boxed{\begin{array}{l}\text{Gross - Pitaevskii equation} \quad \text{(again)} \\ i\hbar\dot{\tilde{\phi}}(x,t) = \hat{H}_0(x)\tilde{\phi}(x,t) + U_0|\tilde{\phi}(x,t)|^2\tilde{\phi}(x,t) \qquad (3.29)\end{array}}$$

*rescale $\tilde{\phi} = \sqrt{N}\phi$

- Same as Eq. (3.24) for $N \simeq N-1$

- A possible quantum-state that justified (3.27) is the many-mode coherent state (2.37).

We assume the S.H.O single-particle basis (1.2) and $|\Psi\rangle = |\alpha_0, \alpha_1 \ldots\rangle$ (see (2.37))

Thus $\underset{\text{state!}}{\langle\Psi|}\underset{\text{field op!}}{\hat{\Psi}}\underset{\text{state}}{|\Psi\rangle} = \langle\alpha_0\alpha_1\ldots|\sum_{k=1}^{\infty}\varphi_k(x)\hat{a}_k|\alpha_0\alpha_1\ldots\rangle$

$$= \langle\alpha_0\alpha_1\ldots|\sum_{k=1}^{\infty}\varphi_k(x)\alpha_k|\alpha_0\alpha_1\ldots\rangle$$

$$= \sum_{k=1}^{\infty}\varphi_k(x)\alpha_k \equiv \tilde{\phi}(x,t=0) \in \mathbb{C}$$

This would also justify factorisation (3.28)

- Due to the use of coherent states, we have an uncertainty in particle number here ($\approx N$ in the mean)

Comments for page 36

- $\left\langle \frac{\partial}{\partial t} \hat{\Psi} \right\rangle = \frac{\partial}{\partial t} \langle \hat{\Psi} \rangle$ since in Heisenberg picture state is time-independent

$\rightarrow$ = acts on...

- $\langle \hat{H}_0(x) \hat{\Psi} \rangle = \langle \Psi | \hat{H}_0(x) \sum_n \varphi_n(x)\, \hat{a}_n | \Psi \rangle$

many body (Fock state)

$$= \hat{H}_0(x) \sum_n \varphi_n(x) \langle \Psi | \hat{a}_n | \Psi \rangle$$

~~about state example~~

$$= \hat{H}_0(x) \underbrace{\langle \Psi | \sum_n \varphi_n(x) \hat{a}_n | \Psi \rangle}_{= \langle \hat{\Psi} \rangle} = \hat{H}_0(x)\, \tilde{\phi}(x)$$

- For coherent state $|\Psi\rangle = |\alpha_0\, \alpha_1 \ldots\rangle$

$$\langle \alpha_0 \alpha_1 \ldots | \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \hat{\Psi}(x) | \alpha_0 \alpha_1 \ldots \rangle$$

$$= \langle \hat{\Psi}^\dagger \rangle \langle \hat{\Psi} \rangle \langle \hat{\Psi} \rangle$$

(excercise, as on p. 36)

- Counterexample where factorisation does not work

[No physics here, just *math* example]

$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|300..\rangle + |200..\rangle\right) \Rightarrow$

$$\langle \Psi | \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \hat{\Psi}(x) | \Psi \rangle = \frac{1}{2} \langle \Psi | \, |\varphi_0(x)|^2 \varphi_0(x)\, \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \, | \Psi \rangle$$

all other pieces of $\sum_{k=1}^{\infty} \varphi_k(x) \hat{a}_k$ than $k=0$ are irrelevant / vanish

$$= \frac{|\varphi_0(x)|^2 \varphi_0(x)}{2} \left(\langle 300| + \langle 200| \right) \left(2\sqrt{3}\, |200\rangle + \sqrt{2} \cdot |100\rangle \right) = \sqrt{3}\, |\varphi_0(x)|^2 \varphi_0(x)$$

But $\langle \hat{\Psi} \rangle = \frac{\varphi_0(x)}{2} \left(\langle 300| + \langle 200| \right) \left(\sqrt{3}\, |200\rangle + \sqrt{2}\, |100\rangle \right) = \frac{\sqrt{3}}{2} \varphi_0(x)$

$\Rightarrow \langle \hat{\Psi}^\dagger \rangle \langle \hat{\Psi} \rangle \langle \hat{\Psi} \rangle = \frac{3\sqrt{3}}{8} |\varphi_0(x)|^2 \varphi_0(x)$

(look check)

3.3.4 Condensate ground state

Start w. ... again

Stationary states of (3.29) evolve as $\tilde{\phi}(x,t) = e^{-i\frac{\mu}{\hbar}t}\,\tilde{\phi}_0(x)$ (3.29b)

Insertion into Eq. (3.29) gives a

time-independent GPE

$$\mu\,\tilde{\phi}_0(x) = \left[\hat{H}_0(x) + U_0\,|\phi_0(x)|^2\right]\phi_0(x) \qquad (3.30)$$

- We renamed $\tilde{\phi}_0 \to \phi_0$ but still now $\sum |\phi_0(x)|^2 = N$
- μ is the chemical potential if ϕ_0 is the ground-state
- as the TISE, (3.30) has ~~...~~ multiple (excited state) solutions [diagram]

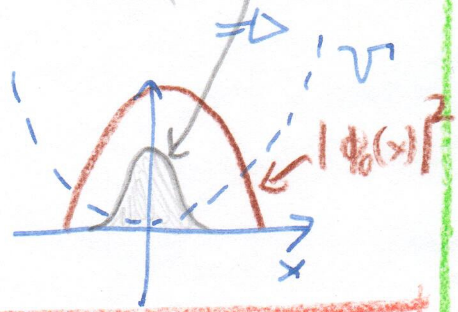
Ground-state solutions: (add imag time)

Example (i): Very strong (repulsive) interactions $U_0 \gg \hbar\omega > 0$ (trap) ⊛ see p. 36C

In that case, neglect $-\frac{\hbar^2}{2m}\nabla^2$ in $\hat{H}_0$

$$\mu\,\phi_0(x) = V(x)\,\phi_0(x) + U_0\,|\phi_0(x)|^2\,\phi_0(x)$$

$$|\phi_0(x)|^2 = \begin{cases} \dfrac{\mu - V}{U_0} & \text{where } > 0 \\ 0 & \text{else} \end{cases}$$



⇒ Wave-function in **Thomas-Fermi approximation**

$$\phi_0(x) = \sqrt{\frac{\mu - V}{U_0}} \quad \text{for } (\mu - V) > 0 \quad (\text{else } = 0) \qquad (3.31)$$

Example (ii): Very weak interactions $|U_0| \ll \hbar\omega$

For $U_0 = 0$, we know harmonic oscillator ground-state ϕ_0 solves (3.30). So

Ansatz:

$$\phi_0(x) = \mathcal{N}\exp\left[-\frac{x^2}{2\sigma(U_0)^2}\right] \qquad (3.32)$$

Determine $\sigma(U_0)$ from variational principle $\delta E = 0$

using $E = \int dx\, \phi_0^*(x)\left[\hat{H}_0 + U_0|\phi_0(x)|^2\right]\phi_0(x)$ Pethick-Smith / or Exercise

Why do we neglect $-\frac{\hbar^2}{2m}\nabla^2$ only?

See: Both rep. Interaction $U_0|\phi_0(x)|^2$ for $U_0>0$

and kinetic energy cause a <u>spread</u> of the wave function

Trap V causes localisation.

Final shape = balance of spread & interaction

For $U_0 n \gg \underbrace{\frac{\hbar\omega}{2}}_{\text{Approx kinetic energy in GS oscillator}}$ k.E. can be neglected relative to interaction.

Example (iii) Any interaction:

Imaginary time method:

Solve (3.29) with $\hbar \to -i\tau$

Imaginary time GPE

$$-\hbar \frac{\partial}{\partial t} \tilde{\phi}(x,\tau) = \left[\hat{H}_0(x) + U_0 \, |\tilde{\phi}(x,\tau)|^2\right] \tilde{\phi}(x,\tau) \qquad (3.32b)$$

subject to constraint $\int_{-\infty}^{\infty} |\tilde{\phi}(x,\tau)|^2 dx \overset{!}{=} N$

This typically rapidly converges to the lowest energy solution $\phi_0(x)$ of (3.30), (almost) regardless of initial state.

heuristic motivation:

Take the (linear) Schrödinger equation ($U_0 = 0$)

Then $\psi(x,t) = \sum_n c_n(0)\, e^{-iE_n t/\hbar} \phi_n(x)$

Replace $t \to -i\tau \Rightarrow$

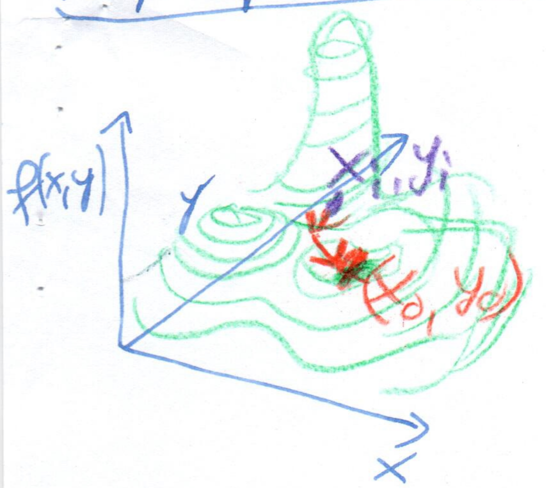
$$\psi(x,\tau) = \sum_n c_n(0)\, e^{-E_n \tau/\hbar} \phi_n(x)$$

$\psi(x,\tau) \to 0$ for $\tau \to \infty$ since all components exponentially decay.

But groundstate component ($n=0$) decays the slowest

$\Rightarrow$ If we enforce $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$, this converges to the groundstate

Imaginary time for $U_0>0$, justification



Let $f(x,y)$ be a 2D function.
To find Minimum & location $f_0(x_0,y_0)$, go opposite to gradient ↑ from some initial test-point (x_i, y_i)

This is called steepest descent method in optimisation

Now consider GP energy functional as ∞-dimensional function

$$f \to E$$
$$(x,y) \to \phi(x)$$

$$E = \int d^3x\, \phi^*(x)\left[H_0(x) + \frac{U_0}{2}|\phi(x)|^2\right]\phi(x)$$

The analog of $\frac{\partial f}{\partial x}$ is $\frac{\delta E}{\delta \phi(x)}$ (functional derivative)

Lets consider ϕ, ϕ^* independent.

$$\frac{\delta E}{\delta \phi^*(x)} = H_0\phi(x) + U_0|\phi(x)|^2\phi(x)$$ (see page

Thus going for a short step $\delta\tau$ into direction of negative gradient

should be ϕ^*??

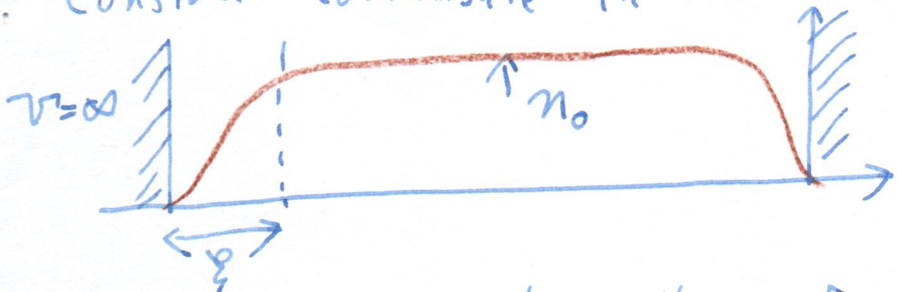
$$\phi(x,\tau+\Delta\tau) - \phi(\tau) = -\left[H_0\phi(x) + U_0|\phi(x)|^2\right]\phi(x)$$

$$\phi(x,\tau+\Delta\tau) - \phi(x,\tau) = -\left[H_0\phi(x,\tau) + U_0|\phi(x,\tau)|^2\right]\phi(x,\tau)$$

⇒ Which is a discrete time-derivative version of Eq. (3.32b)

3.3.5. Condensate healing length

Consider condensate in hard box, for large U_0 — $|\Phi_0(x)|^2$



At edges ψ has to vanish due to boundary conditions.

- Far away from edges, Thomas-Fermi (3.31) gives $n_0 = \frac{\mu}{U_0}$ (3.33)
- Near edge, cannot neglect kinetic term

Rewrite (3.30) as

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi_0(x) + U_0\left[|\Phi_0(x)|^2 - n_0\right]\Phi_0(x) = 0$$

Define $\phi_0(x) = \sqrt{n_0}\, f\left(\frac{x}{\xi}\right) \Rightarrow$

$$-\frac{\hbar^2}{2mU_0 n_0 \xi^2} f''\left(\frac{x}{\xi}\right) + \left[f^2 - 1\right] f = 0 \qquad (3.34)$$

This Eqn becomes scale-free if

$$\boxed{\text{healing length} \quad \xi = \frac{\hbar}{\sqrt{2mU_0 n_0}} \qquad (3.35)}$$

- this is the shortest scale on which the BEC can respond to perturbations in the bulk

3.3.6. Hydrodynamic equations and vortices

Let us rewrite $\phi(x,t) = \underbrace{\sqrt{\rho(x,t)}}_{\text{amplitude}}\, e^{i\overbrace{\varphi(x,t)}^{\text{phase}}}$ (3.36) $\quad \phi \in \mathbb{C},\ \rho, \varphi \in \mathbb{R}$

These have the interpretation

atomic density $\rho = |\phi|^2$

and flow velocity $\vec{v} = \frac{\hbar}{m}\vec{\nabla}\varphi$ (3.37)

To see that this makes sense we insert (3.36) into Eq. (3.29) split into Re() & Im() and derive

Hydrodynamic equations for a BEC

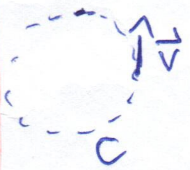
continuity equation $\dfrac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\cdot\vec{v})$ (3.38)

"Bernoulli's" equation $m\dfrac{d\vec{v}}{dt} = -\vec{\nabla}\left[P_q + \tfrac{1}{2}m\vec{v}^2 + U\cdot\rho + V(x)\right]$ (3.39)

with quantum pressure term $P_q = \dfrac{-\hbar^2\nabla^2\sqrt{\rho}}{2m\sqrt{\rho}}$ (3.40)

- Whenever P_q is small, can think of BEC as "fluid"
- quantum nature still has interesting consequence such as

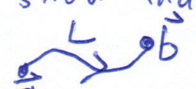
Quantisation of circulation



$$\underbrace{\oint_C \vec{v}\cdot d\vec{x}}_{\text{cirulation}} = (2\pi n)\frac{\hbar}{m} = n\underbrace{\left(\frac{h}{m}\right)}_{\text{circulation quantum}} \quad (3.41)$$

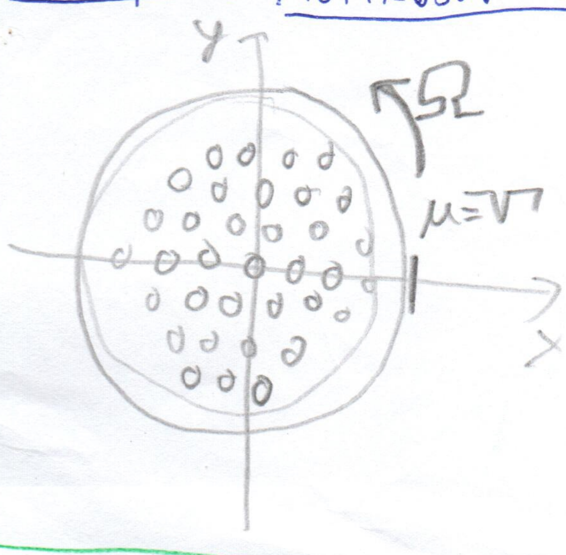
$n \in \mathbb{N}_0$ winding number

- Proof: show that $\int_L \vec{v}\cdot d\vec{x} = \frac{\hbar}{m}\left[\varphi(\vec{b}) - \varphi(\vec{a})\right]$ for a non-closed loop



then (3.41) follows because the phase at $\vec{x}$ has to be unique $\Rightarrow$

Example: Abrikosov-lattice:



A BEC brought to high circulation state $n \gg 1$ forms an Abrikosov-lattice of n Vortices

each vortex-core has $\rho = 0$, due to undefined phase:

7. Condensate excitations

What happens to a stationary state (3.29b) if it is slightly perturbed? Look for <u>periodic (eigenmode)</u> solutions with <u>Ansatz:</u>

$$\boxed{\text{Perturbed BEC:}\quad \phi(x,t) = e^{-i\mu t/\hbar}\left[\phi_0(x) + u(x)e^{-i\omega t} - v^*(x)e^{i\omega t}\right] \quad (3.42)}$$

- We need to include $e^{i\omega t}$ AND $e^{-i\omega t}$ because (3.29) <u>couples</u> ϕ and ϕ^*
- Insert (3.42) into (3.29), use (3.30), seperately consider coefficients of $e^{i\omega t}$ and $e^{-i\omega t}$. Get:

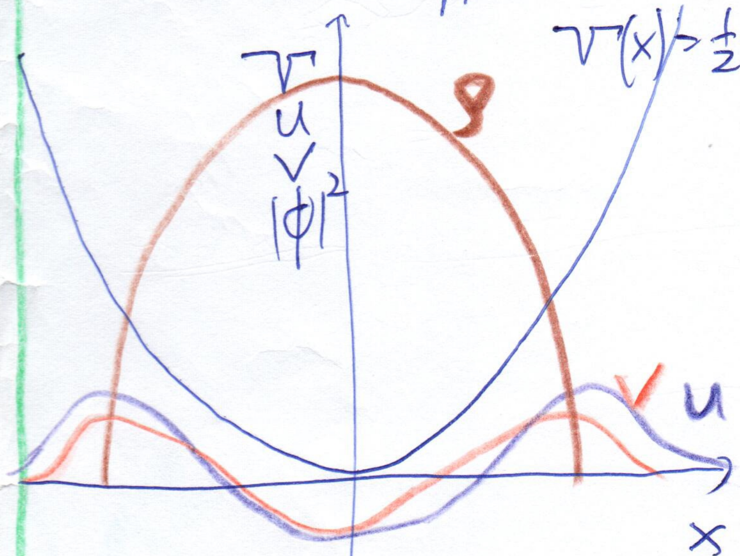
Bogoliubov equation for <u>elementary excitations</u> of BEC

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x) + 2U_0|\phi_0(x)|^2 - \mu - \hbar\omega_m\right]u(x) - U_0\,\phi_0(x)^2\,v(x) = 0$$
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x) + 2U_0|\phi_0(x)|^2 - \mu + \hbar\omega_m\right]v(x) - U_0\,\phi_0^*(x)^2\,u(x) = 0 \quad (3.43)$$

ω_m = Mode frequency

Example: Breathing mode of harmonically trapped BEC

Find one mode with $\phi_0, u, v \in \mathbb{R}$ as shown left.



$$|\phi(x,t)|^2 \approx |\phi_0(x)|^2 + 2\phi_0(u-v)\cos(\omega t) + O(u^2, v^2)$$

This will do <u>breathing oscillations</u> from competition of trapping and repulsive interactions.

<u>Breathing frequency</u> (for large U_0)

$$\omega_m = \sqrt{5}\,\omega_{trap}$$