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2.4. Coherent states

2.4.1. Coherent harmonic oscillator states

Q: What is the "most classical" type of oscillation we can get in the quantum H.O.

Answer: Define

Coherent state:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \exp[\alpha \hat{b}^+]|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle \qquad \alpha \in \mathbb{C} \qquad (2.34)$$

also: $|\alpha\rangle = D(\alpha)|0\rangle \qquad D(\alpha) = e^{\alpha \hat{a}^+ - \alpha^* \hat{a}}$

Displacement operator

- Here $\hat{b}^+$ is from (1.4) Ladder operator.
- Coherent states are not eigenstates of $\hat{H}_{SHO}$, uncertain energy/number of oscillator quanta

Properties of coherent states

$$\hat{b}|\alpha\rangle = \alpha|\alpha\rangle \qquad \hat{b}^+|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle \qquad (2.35)$$

$$\langle\alpha|\hat{b}^+ = \langle\alpha|\alpha^*$$

$$\langle\alpha|\alpha'\rangle = \exp\left[\alpha^*\alpha' - \frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2}\right] \quad \text{(Not orthogonal)}$$

$$\langle\alpha|\alpha\rangle = 1$$

$$\mathbb{1} = \int d\alpha\, |\alpha\rangle\langle\alpha|$$

- Coherent state is right-eigenstate of destruction operator

Example: Oscillation of coherent state (Negele Orland p. 39)

What is the meaning of α? Let $\alpha_0 \in \mathbb{R}$

Convert $\hat{b}|\alpha_0\rangle = \alpha_0|\alpha_0\rangle$ to position basis:

$$\langle x|\hat{b}|\alpha_0\rangle = \alpha_0 \underbrace{\langle x|\alpha_0\rangle}_{\equiv \tilde{\alpha}_0(x)} \xrightarrow[(1.4)]{\text{using}} \frac{\partial}{\partial x}\tilde{\alpha}_0(x) = \left(-\frac{m\omega}{\hbar}x + \sqrt{\frac{2m\omega}{\hbar}}\alpha_0\right)\tilde{\alpha}_0(x)$$

$$\overset{\text{solve DE}}{\Rightarrow} \tilde{\alpha}_0(x) = C \exp\left[-\frac{(x-\alpha_0')^2}{2\sigma^2}\right]$$

$$\sigma = \sqrt{\frac{\hbar}{m\omega}} \qquad \alpha_0' = \sqrt{2}\,\sigma\alpha_0$$

Proof of (2.35) Let $|\bar{\alpha}\rangle = e^{\frac{|\alpha|^2}{2}}|\alpha\rangle$

$$\hat{b}|\bar{\alpha}\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, \hat{b}|\varphi_n\rangle = \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\,|\varphi_{n-1}\rangle$$

$n \mapsto n+1$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{\alpha^{n+1}}{\sqrt{(n+1)!}} \sqrt{(n+1)}}_{\alpha \frac{\alpha^n}{\sqrt{n!}}}\, |\varphi_n\rangle = \alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle = \alpha|\bar{\alpha}\rangle \quad \square$$

$$\hat{b}^+|\bar{\alpha}\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, \hat{b}^+|\varphi_n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}\,|\varphi_{n+1}\rangle$$

$$= \sum_{n=0}^{\infty} \frac{\frac{\partial}{\partial\alpha}\alpha^{n+1}\sqrt{n+1}}{\sqrt{n!}\,(n+1)} |\varphi_{n+1}\rangle = \sum_{n=0}^{\infty} \frac{\frac{\partial}{\partial\alpha}\alpha^{n+1}}{\sqrt{(n+1)!}} |\varphi_{n+1}\rangle$$

$$= \frac{\partial}{\partial\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle = \frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle = \frac{\partial}{\partial\alpha}|\bar{\alpha}\rangle \quad \square$$

(Rest follows fr. ~~d~~ product rule)

Time evolution: $\hat{H}_0 |\varphi_n\rangle = E_n |\varphi_n\rangle$ $E_n = \hbar\omega(n+\frac{1}{2})$
of this state

$$\Rightarrow |\alpha(t)\rangle = \sum_n \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\omega(n+\frac{1}{2})t} |\varphi_n\rangle = e^{-i\frac{\omega}{2}t} |\underbrace{\alpha_0 e^{-i\omega t}}_{\to \alpha(t)}\rangle$$

group $\alpha_0^n e^{-i\omega n t} = (\alpha_0 e^{-i\omega t})^n$

Can show after some fiddling

$$|\tilde{\alpha}(x,t)|^2 = c' \exp\left[-\frac{(x-\alpha_0' \cos(\omega t))^2}{\sigma^2}\right]$$

We thus always have a groundstate shaped Gaussian oscillating in the potential with amplitude α_0'

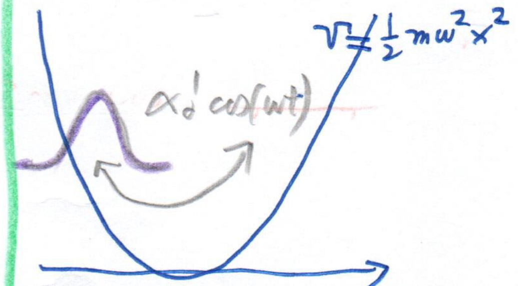


Fig1.

In phase space/complex plane

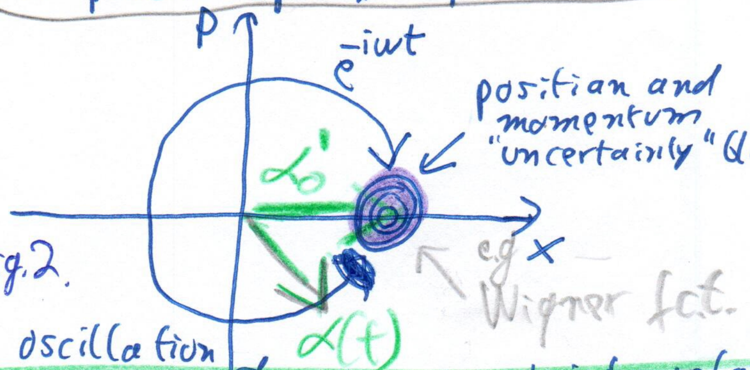


Fig.2.

Closest we can get to classical oscillation despite uncertainty relations

2.4.2. Coherent many-body states

Due to the ~~same~~ identical properties of ladder operators $\hat{b}$ and $\hat{a}, \hat{c}$, we can equally define

Many-body coherent state: (Bosons)

$$|\alpha\rangle = \exp[\hat{a}_m^+ \alpha]|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad \alpha \in \mathbb{C} \quad (2.36)$$

Fock-state $|n\rangle$: occupation of mode $|\varphi_m\rangle$

- this now describes a superposition of different occupation-numbers (Fock states) of single-body mode $|\varphi_m\rangle$
- all properties of (2.35) apply

Prior to Fig. 2 in example above (page 24)

Classically we have the idea of phase-space (x, p)
Quantum-mechanically $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$ $\Rightarrow$ particle cannot have a fixed phase-space-coordinate.

We can still represent a quantum state $\psi(x)$ in phase-space, using

Wigner distibution

$$W(x,p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \psi^*(x+y)\, \psi(x-y)\, e^{2ipy/\hbar}\, dy \quad (2.35b)$$

- Properties

$$\int_{-\infty}^{\infty} dp\, W(x,p) = |\psi(x)|^2 \quad \text{position-space distribution}$$

$$\int_{-\infty}^{\infty} dx\, W(x,p) = |\tilde{\psi}(p)|^2 \quad \text{momentum-space distribution}$$

- $W(x,p)$ is a quasi-probability distribution
(Means we can get some expectation values by integrating over it, but it may have regions with $W(x,p) < 0$
- The interpretation is that drawing $(W(x,p))$, non-zero elements locate a quantum-state in phase-space

$\hookrightarrow$ Fig. 2 on page 24

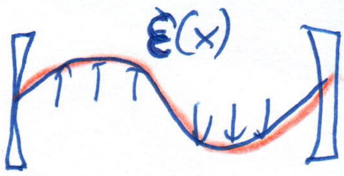
Wigner function for number state (see Fig. 3 p. 25)

$$\chi_W(\lambda, \lambda^*) = \mathrm{Tr}\{\hat{\rho} \exp(\lambda \hat{a}^\dagger - \lambda^* \hat{a})\}$$

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda\, \exp(-\lambda\alpha^* + \lambda^*\alpha)\, \chi_W(\lambda, \lambda^*) \quad (2.35c)$$

Example: Laser

Consider a single-mode photon field at frequency ω



$$\hat{H} = \hbar\omega\, \hat{a}^{+}\hat{a} \qquad \text{just as for oscillator}$$

Electric field (c.f. Example C page 19) Check w. Walls & Milburn

$$\hat{E}(x,t) = \mathcal{E}(x,t)\,\hat{a} + \text{H.c.}$$

Taking the expectation value in the coherent state $|\alpha(t)\rangle$ (see example page 24), we can show (excercise)

$$\langle\alpha(t)|\,\hat{E}(x,t)\,|\alpha(t)\rangle = 2\,\mathrm{Re}\{\mathcal{E}(x,t)\underbrace{\alpha_0 e^{-i\omega t}}_{\alpha(t)}\}$$

Thus here, the complex number $\alpha(t)$ characterizes amplitude and phase of the oscillating electric field:

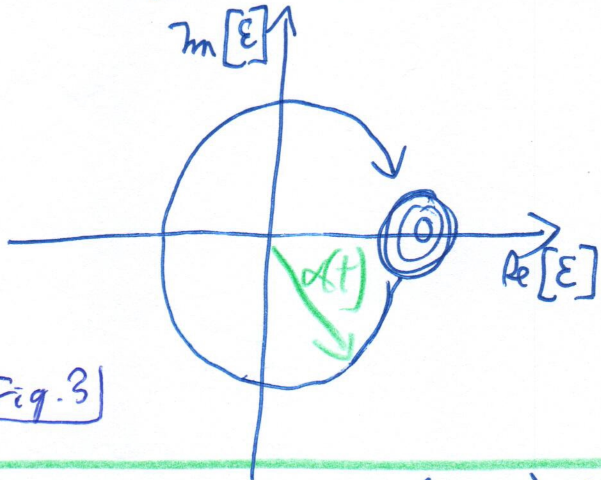


Fig. 3

We can combine states (2.36) into

many-mode coherent states (Bosons)

$$\hat{a}_k|\vec{\alpha}\rangle = \alpha_k|\vec{\alpha}\rangle \qquad \vec{\alpha} = \{\alpha_1 \dots \alpha_N\} \quad \alpha_k \in \mathbb{C} \qquad (2.37)$$

which exhibit one coherent amplitude α_k for each single particle basis state k

- The slightly messy formal decomposition of (2.37) into Fock states is

$$|\vec{\alpha}\rangle = e^{-\sum_k \frac{|\alpha_k|^2}{2}} \sum_{\substack{n_1,n_2,n_3\\ \dots n_N}} \frac{\alpha_1^{n_1}\alpha_2^{n_2}\cdots\alpha_N^{n_N}}{\sqrt{n_1!}\sqrt{n_2!}\cdots\sqrt{n_N!}}\,|n_1\, n_2\, n_3 \dots n_N\rangle \qquad (2.38)$$

2.4.3. Fermionic coherent states

If we assume a definition like (2.37) for Fermionic operators we run into trouble:

$$\{\hat{a}_k, \hat{a}_l\}|\vec{\alpha}\rangle = (\alpha_k\alpha_l + \alpha_l\alpha_k)|\vec{\alpha}\rangle \overset{!}{=} 0 \quad \left(\text{since } \{\hat{a}_k, \hat{a}_l\} = 0\right)$$

For two non-zero complex numbers $\underbrace{\alpha_k\alpha_l + \alpha_l\alpha_k}_{=2\alpha_l\alpha_k} \neq 0$ of course.

Solution:

$\boxed{\text{Define anti-commuting complex numbers called } \underline{\text{Grassmann}} \text{ - numbers}}$

- based on these we can also use the coherent state concept for Fermions. Mainly useful for Fermionic path integrals.
- Not further used in this lecture.