3

2.3. QUANTUM FIELDS

Let us ~~[illegible]~~ expand our assembly of the second quantized Hamiltonian in (2.21) again

$$\hat{H} = \sum_{nm} \langle \varphi_n | \hat{A} | \varphi_m \rangle \hat{a}^+_n \hat{a}_m + \sum_{\substack{nm\\ lk}} \langle \varphi_n \varphi_m | \hat{B} | \varphi_l | \varphi_k \rangle \hat{a}^+_n \hat{a}^+_m \hat{a}_l \hat{a}_k$$

Using the position space representation of $|\varphi_n\rangle$, this becomes

$$\hat{H} = \sum_{nm} \int dx\, \varphi^*_n(x)\, \hat{A}(x) \varphi_m(x)\, \hat{a}^+_n \hat{a}_m + \sum_{\substack{nm\\ lk}} \int dx \int dy \;\times$$

$$\varphi^*_n(x)\, \varphi^*_m(y)\, \hat{B}(x,y) \varphi_l(x)\, \varphi_k(y)\, \hat{a}^+_n \hat{a}^+_m \hat{a}_l \hat{a}_k$$

We now lump together the position-space single particle basis functions $\varphi_n(x)$ and operators $\hat{a}_n$ into the

$$\boxed{\text{Field-operator:} \quad \hat{\Psi}(x) = \sum_n \varphi_n(x)\, \hat{a}_n \qquad (2.26)}$$

Using this notation, the Hamiltonian is

$$\hat{H} = \int dx\, \hat{\Psi}^+(x)\, \hat{A}(x)\, \hat{\Psi}(x) + \int dx \int dy\, \hat{\Psi}^+(x)\, \hat{\Psi}^+(y)\, B(x,y)\, \hat{\Psi}(x) \hat{\Psi}(y) \qquad (2.27)$$

For the same case as (2.23) we have:

<u>Hamiltonian for N particles in a 1D harmonic trap (with interactions)</u>

$$\boxed{\hat{H} = \int dx\, \hat{\Psi}^+(x) \underbrace{\left[-\frac{\hbar^2}{2m} \nabla^2_x + V(x)\right]}_{\equiv \hat{H}_0} \hat{\Psi}(x) + \frac{1}{2} \int dx \int dy\, \hat{\Psi}^+(x)\, \hat{\Psi}^+(y)\, U(x-y)\, \hat{\Psi}(x) \hat{\Psi}(y) \qquad (2.28)}$$

- Field operator is also simply the annihilation operator for the <u>position-basis</u>. Think of it as <u>annihilating a particle</u> at position "x".

- all three descriptions (2.19), (2.21), (2.28) are fully equivalent, which is "best" depends on problem. $\boxed{++}$

- Using (2.17) and ~~[illegible]~~ $\sum_n \varphi_n(x)\varphi_n^*(y) = \delta(x-y)$ we can show

Commutation relations for field operators:

$$\text{Bosons:}\quad [\hat{\Psi}(x), \hat{\Psi}^{\dagger}(y)] = \delta(x-y), \quad [\hat{\Psi}(x), \hat{\Psi}(y)] = 0 \tag{2.29}$$

$$\text{Fermions:}\quad \{\hat{\Psi}(x), \hat{\Psi}^{\dagger}(y)\} = \delta(x-y), \quad \{\hat{\Psi}(x), \hat{\Psi}(y)\} = 0$$

2.3.1. Examples of Quantum Fields

Advantages/strengths of quantum field concept: (but do not need it, see $\boxed{++}$)

- Naturally deals with particle creation/destruction and conversion. Different Fock-states (2.14) can be viewed as different excited states of the underlying field.
- Formulated in time & space $(t, \vec{x})$, can e.g. naturally address spatial & temporal coherence properties (e.g. see chapter 3)
- can conveniently formulate Lorentz-invariant (relativistic) theories, and take care of causality

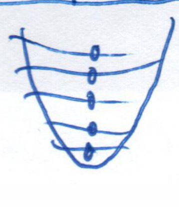
Example A: Harmonically trapped dilute Bose gas:

$$\hat{\Psi}(x) = \sum_n \varphi_n(x)\, \hat{a}_n$$

$\uparrow$ SHO Modes

- typically atom number conserved and non-relativistic
- Field operator useful to describe coherence and condensation
- Atom-number may fluctuate if part of system external

Example B: — " — Fermi gas



Example C: Quantized light field (QED, quantum optics)
(electric)

$$\hat{\vec{E}}(\vec{r},t) = \sum_k \vec{\varepsilon}_k \, \mathcal{E}_k \, \hat{a}_k \, e^{-i\omega_k t + i\vec{k}\vec{r}} + \text{H.c.}$$

("$\hat{\varphi}$") polarisation — Amplitude — plane-wave (photon-mode)

wave/particle duality

Example D: Relativistic spin-$\frac{1}{2}$ Field (e.g. quarks/electrons)

$$\hat{\psi}_{\boxed{\alpha}}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3 2p_0} \Big(e^{-ipx} \underbrace{u_{\boxed{\alpha}}(p,s)}_{\text{spinor}} \hat{a}(p,s) + e^{ipx} v_\alpha(p,s) \hat{a}^\dagger(p,s) \Big)$$

spin index — 4-vector $(t,\vec{x})$ — particles — anti-particles

Example D: Non-relativistic electron gas in condensed-matter

$$\hat{\psi}_\sigma(x) = \sum_n \int dk \, \hat{a}_{nk} \, \underbrace{u_{nk}(r)}_{\text{Bloch function}} \underbrace{e^{ikx}}_{\text{plane wave}} \underbrace{\chi_\sigma}_{\text{spin}} \quad + \text{c.c.} \; (?)$$

n-Band index

R

$u_{nk}(r+R) = u_{nk}(r)$

2.3.2. Dynamics of quantum fields

Here The field operators are mainly a way to re-write the Hamiltonian. Much of ~~the~~ the usual methodology of quantum mechanics can be applied as before.

E.g. Consider the Heisenberg-picture* for the field operator in (2.28): $i\hbar\dot{\hat{\psi}} = [\hat{\psi}, \hat{H}]$:

Heisenberg eqn for Field-operator:

$$i\hbar\dot{\hat{\psi}}(x,t) = \hat{H}_0 \hat{\psi}(x,t) + \int d^3y \, \hat{\psi}^\dagger(y,t) U(x-y) \hat{\psi}(y,t) \hat{\psi}(x,t) \qquad (2.30)$$

- We have made use of the commutation relation (2.29)
- We shall begin exploring BEC from here in chapter 3

*If unfamiliar, pls revise Schrödinger picture, Heisenberg picture, ...

Example: Non-interacting evolution of ~~Bose~~ Atom-field (as in 2.23 for $U=0$)

$$\hat{\Psi}(x) = \sum_n \varphi_n(x)\,\hat{a}_n \quad \text{into} \quad (2.30)$$

$$i\hbar \sum_n \varphi_n(x)\,\dot{\hat{a}}_n = \hat{H}_0\left(\sum_n \varphi_n(x)\,\hat{a}_n\right)$$

$$= \sum_n \underbrace{\hat{H}_0 \varphi_n(x)}_{= E_n \varphi_n(x)}\,\hat{a}_n$$

$$\int dx\, \varphi_m^*(x) \ldots \qquad i\hbar\,\dot{\hat{a}}_m = E_m \hat{a}_m \qquad \hat{a}_m(t) = \hat{a}_m(0)\, e^{\frac{-iE_m t}{\hbar}}$$

$$\hat{\Psi}(x,t) = \sum_n \varphi_n(x)\, e^{-\frac{iE_n t}{\hbar}}\,\hat{a}_n(0) \qquad (2.30b)$$

The number of cases where (2.30) can be solved is limited. But we also still have:

Time-evolution operator:

$$\hat{U}(t, t_0) = T\left[\exp\left[-i \int_{t_0}^{t} dt' \hat{H}(t')\right]\right] \qquad (2.31)$$

- Evolves a (many-body) quantum state in time

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle$$

- $\hat{H}$ & initial/final states can be written ~~with~~ using field operators.

~~Transition amplitudes~~

- We can move to the interaction picture to replace $\hat{H}(t')$ with some (weaker) interaction $\hat{V}(t')$ in (2.31)
- Then expand exp in a power series → time-dependent perturbation theory, Feynman diagrams (not here).

See Bruus and Flensberg *

2.3.3. Observables and Green's functions

(1) As usual in QM, all physical observations related to a quantum field ~~are~~ can be ~~expe~~ written as expectation value of an operator

(2) In 2.2.1. we showed how all operators can be expressed by creation- (destruction-) operators $\hat{a}^+(\hat{a})$

(3) These in turn can all be expressed through field operators.

All up, a huge list of phenomena can be understood from the /Correlation functions

Green's function: Roughly of the form

$$G^{(n)}(x_1 t, \ldots x_n t \,|\, x_1' t' \ldots x_n' t') \qquad (2.32)$$
$$= \left\langle \hat{\psi}^+(x_n' t') \ldots \hat{\psi}^+(x_1' t') \hat{\psi}(x_1, t) \ldots \hat{\psi}(x_n, t) \right\rangle$$

[there are lots and lots of alternative definitions

2.3.4. Spin-statistics theorem

This really belongs in the realm of relativistic quantum mech. or particle physics, but we could not resist sketching it here

you know that:

$$\text{Half-integer spin particles} = \text{Fermions} \quad (s = \tfrac{1}{2}, \tfrac{3}{2}, \text{ etc.})$$
$$\text{Integer spin particles} = \text{Bosons} \quad (s = 0, 1, 2 \text{ etc.}) \qquad (2.33)$$

(Spin-statistics theorem)

This follows necessarily from causality and Lorentz invariance.

Rough sketch of proof:

- Hamiltonian density $\hat{H} = \int d^4x \, \hat{\mathcal{H}}(x)$ $\quad (x = 4 \text{ vector } (t, \vec{x}))$
  must obey $[\hat{\mathcal{H}}(x), \hat{\mathcal{H}}(y)] = 0$ for $(x-y)^2 = c^2\Delta t^2 - \Delta\vec{x}^2 < 0$ Ⓐ

  replace with fields directly

  (space-like separation)

  (This means space-like separated events cannot affect each other.

- quantum fields ~~[illegible]~~ obey specific transformation laws under Lorentz-transformations $\Lambda \leftarrow 4\times4$ matrix, ~~[illegible]~~ depending on spin of field

  Needed for $(x-y) \leftrightarrow (y-x)$

  $$U(\Lambda)\hat{\varphi}^{+}(x)\,U^{-1}(\Lambda) = \hat{\varphi}^{+}(\Lambda x) \qquad (\text{spin zero})$$
  $$U(\Lambda)\hat{\varphi}_s^{+}(x)\,U^{-1}(\Lambda) = \sum_{s'} \underbrace{D_{ss'}(\Lambda^{-1})}_{\text{representation matrix}} \hat{\varphi}_{s'}^{+}(\Lambda x) \qquad (\text{spin e.g. } \tfrac{1}{2})$$

- It turns out that Ⓐ works out if

  $$[\hat{\varphi}(x), \hat{\varphi}^{+}(y)] = 0 \quad \text{for } (x-y)^2 < 0 \qquad (\text{integer spin})$$
  $$\{\hat{\varphi}_s(x), \hat{\varphi}_{s'}^{+}(y)\} = 0 \quad \text{—\,||\,—} \qquad (\text{half-integer spin})$$

  $\Rightarrow$ see Eq. (2.20)

2.3.5. Notation overview-I:

We have now introduced a lot of different but similar symbols for single- vs. many-body states and operators. We will attempt to stick to the following notation:

$\|w_n\rangle$, $\|\varphi_n\rangle$, $w_n(\vec{x})$, $\varphi_n(\vec{x})$	—	single particle basis(es) and their position representations
$\psi(\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots)$, $\|\psi\rangle$	—	Many-body state and its (1st quantized) position representation
$\hat{\psi}(\vec{x})$	—	Field operator
$\hat{b}$, $\hat{b}^+$	—	harmonic oscillator ladder operator
$\hat{a}_n, \hat{a}_n^+, \hat{c}_n, \hat{c}_n^+$	—	creation and annihilation operators for various basises ($\|w_n\rangle$, $\|\varphi_n\rangle$...)

Quiz review if too fast