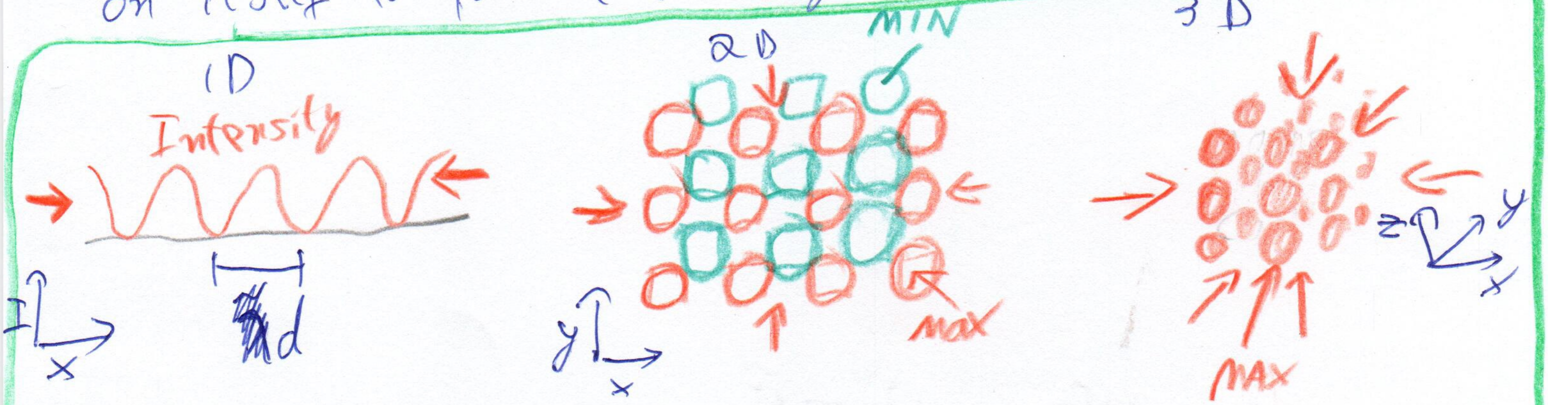
13

5.2. Lattice-models

We have mentioned few times, the similarities between an electron gas in a crystal lattice and a cold atomic gas in an optical lattice. More details on the latter

5.2.1. Optical lattices

Consider a coherent laser beam which is back-reflected on itself to form a standing wave in



• We obtain an intensity pattern for the light that can be written as,

$$I(x) = I_0 \cos^2(k_L x) \qquad (5.7)$$

distance between sites $d = \frac{\lambda_L}{2}$ $\quad k_L = \frac{2\pi}{\lambda_L}$

λ_L = laser wavelength

We can calculate the energy shift of an atom due to exposure to the rapidly varying E-field of the laser. (AC-Stark shift). This shift turns $I(x)$ into a spatial potential

$$V(x) = -\frac{1}{2}\alpha(\omega)\langle \mathcal{E}(t)^2\rangle_t \qquad (5.8)$$

atomic polarizability; laser frequency; light intensity; time average

We find $\alpha(\omega) < 0$, just above an atomic resonance (blue detuned) $\Rightarrow V > 0$
$\alpha(\omega) > 0$, just below — " — (red detuned) $\Rightarrow V < 0$
(See also PHY 402, Assignment 4)

$\Rightarrow$ Optical Lattice potential

$$V(x) = V_0 \cos^2(k_L x) \qquad (5.9)$$

$V_0 > 0$ OR $V_0 < 0$

5.2.2. Bose-Hubbard Model

You had shown in the mid-term exam, how, starting from (3.25) [$\hat{H}$ for Bose gas in form with $\hat{\Psi}$], we can derive <u>Bose-Hubbard Hamiltonian:</u>

$$\hat{K} = \hat{H} - \mu\hat{N} = \sum_m J\left(\hat{a}^+_{m+1}\hat{a}_m + \hat{a}^+_{m-1}\hat{a}_m\right) + \frac{U}{2}\hat{n}_m(\hat{n}_m - 1) - \tilde{\mu}\hat{n}_m \qquad (5.10)$$

- $\hat{a}^+_m$ creates an atom "on site m"

- J allows tunneling / hopping from site to site
- U are repulsive on-site interactions $\hat{n}_m = \hat{a}^+_m\hat{a}_m$
- re-defined wrt. exam $\tilde{\mu} = \mu - E$ in the chemical potential (E: on site energy)

Let us try to find ground-states of $\hat{K}$

two simple cases:

A $J = 0$, no tunneling. $[\hat{K}, \hat{n}_m] = 0 \Rightarrow$ We can write Eigenstates as Fock-states $|\vec{N}\rangle$. Since all sites are equivalent pick $|\vec{N}_0\rangle = |M, M, \dots M\rangle$, i.e. exactly <u>$M$ Bosons</u> per site

$$\langle\vec{N}_0|\hat{K}|\vec{N}_0\rangle = N_{sites}\cdot\left(\frac{U}{2}M(M-1) - \tilde{\mu} M\right)$$

This is minimized by $m = \frac{\tilde{\mu}}{U} + \frac{1}{2}$. Since m has to be integer

For $M - 1 < \frac{\tilde{\mu}}{U} < M$ we have exactly M Bosons per site.

<u>Mott-insulating state:</u>

$$|\Psi_{mott}\rangle = \sum_m \frac{(\hat{a}^+_m)^M}{\sqrt{M!}}|0\rangle \qquad (5.11)$$

- Mott-insulator: Material should conduct fr. Band-theory (single particle), but does not due to $e^- - e^-$ interactions.
- gapped-excitations

(B) $U=0$, no interactions

Solve single-particle Hamiltonian, lowest state

$$|\varphi_0\rangle = \frac{1}{\sqrt{N}} \sum_m \hat{a}_m^+ |0\rangle \qquad (5.12)$$

(particle fully de-localized on lattice)

- We know at $T=0$ we will have a BEC condensed in $|\varphi_0\rangle$ and can use mean-field theory. $\langle\hat{\psi}\rangle = \phi \neq 0$

$$\boxed{\text{Superfluid state: } \quad |\Psi_{BEC}\rangle = \underbrace{N}_{\text{normalis. factor}} \left(\sum_m \hat{a}_m^+ \right)^M |0\rangle \qquad (5.12)}$$

Let us translate here $\langle\hat{\psi}\rangle \neq 0$ into $\langle\hat{a}_m\rangle \neq 0$ and set

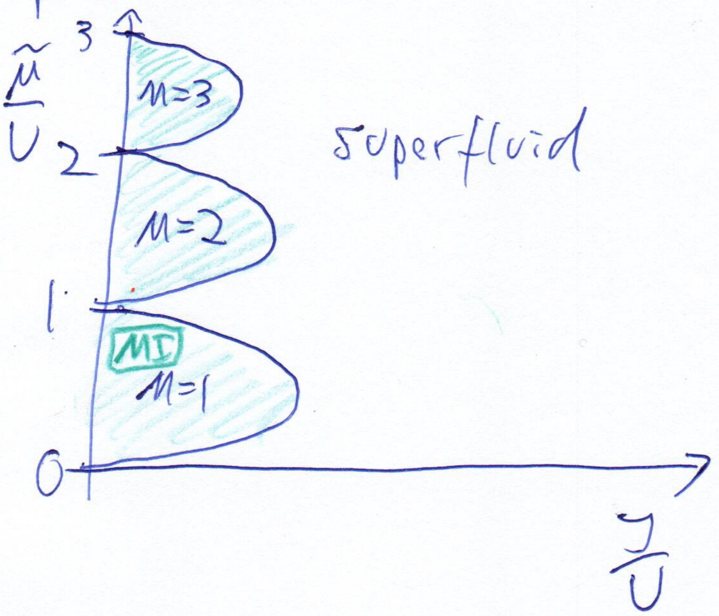
$$\hat{a}_m = \underset{\in \mathbb{C}}{\alpha} + \underbrace{\delta\hat{a}_m}_{\text{operator, (Not small!!!)}}$$

(α = const on all sites ⇒ homogeneous system) & then rename $\delta\hat{a}_m \to \hat{a}_m$

We obtain mean-field Hamiltonian (linear in $\hat{a}_m, \hat{a}_m^+$)

For both $U \neq 0$, $J=0$ require more complicated analysis. Result:

Phase-diagram for Superfluid / Mott-insulator
Quantum phase transition



- This is called quantum phase transition, because everything happens at $T=0$.

From (5.11), (5.12) can calculate inter-site coherence

$$g_{m,m+1} = \langle \hat{a}_m^+ \hat{a}_{m+1} \rangle .$$

We find $g=0$ for MI (5.11) and $g \neq 0$ for superfluid (5.12) (Exercise).

This means atoms from different sites interfere in the superfluid after T.O.F expansion, but not in the Mott-insulator

$\Rightarrow$

clear experimental signature, see
Greiner et. al, Nature 415 (2002) 39.