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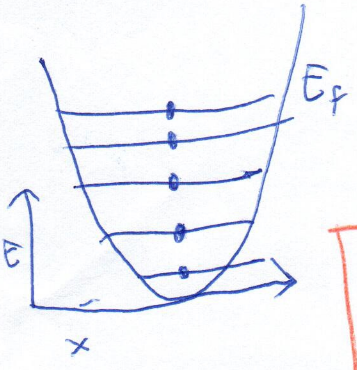
# 4.7. Trapped atomic Fermi gases




Non-interacting ground-state: All single-particle states $|\varphi_n\rangle$ up to $E = E_F$ are filled with exactly one atom. (or $2S+1$ if we consider them spin $S$)

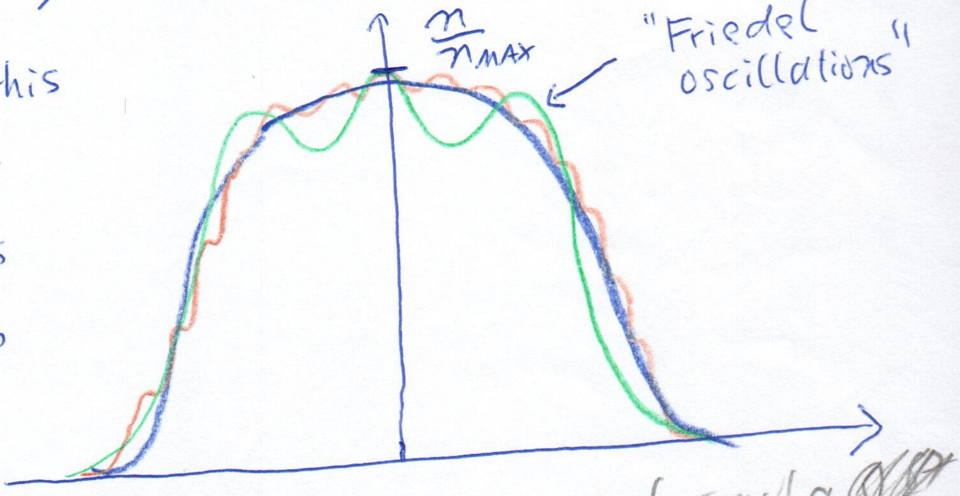
> Define Fermi-sea state
>
> $$|FS\rangle_N = \prod_{\substack{n \\ E_n < E_F(N)}} \hat{a}_n^{+} |0\rangle \qquad (4.14)$$
>
> $N$ = atom-number

Using Fermi-field operator $\hat{\Psi}(x) = \sum_n \varphi_n(x)\, \hat{a}_n$, ← Eq. (1.2)

we obtain a total density

$$n(x) = \langle FS | \hat{\Psi}^{+}(x) \hat{\Psi}(x) | FS \rangle \underset{\text{excercise}}{=} \sum_n |\varphi_n(x)|^2 \qquad (4.15)$$

This looks like this




few atoms

more atoms

many atoms → approaches smooth function. which?

↗ Cristoffel - Darboux - formula

## 4.7.1. Thomas-Fermi-approximation

We again use the Thomas-Fermi approximation to find shape, in a slightly different formulation.

Let's assume a large gas, so that we can locally use the homogeneus results from 4.1.

(but now using a density $\frac{N}{V} \to n(\underline{r})$

From (4.1), (4.3), we can then define local Fermi wavenumber / momentum via

$$n(r) = \frac{k_F^3(r)}{6\pi^2} \qquad \varepsilon_F(r) = \frac{\hbar^2 k_F^2(r)}{2m} \qquad (4.16)$$

we ignore spin here

The equilibrium density must be such, that adding one more atom has the same energy everywhere

$$\underbrace{\frac{\hbar^2 k_F^2(r)}{2m}}_{\text{added on Fermi surface}} + \underbrace{V(r)}_{\text{trap}} = \mu \qquad (4.17)$$

solving for $n(r)$ gives Thomas-Fermi profile of Fermi-gas

$$n(r) = \frac{1}{6\pi^2}\left\{\frac{2m}{\hbar^2}\underbrace{\left[\mu - V(r)\right]}_{\text{if } >0,\ n(r)=0 \text{ else}}\right\}^{3/2} \qquad (4.18)$$

- this gives the blue line on page 66 ①
- Note for BEC we had $(\mu - V(r))$

We can extend this local ~~/~~ semi-classical / like WKB approach to momentum and finite temperature with distribution function

$$f(\vec{r}, \vec{p}) = \frac{1}{\exp\left[\beta\left(\frac{\vec{p}^2}{2m} + V(\vec{r}) - \mu\right)\right] + 1} \qquad \beta = \frac{1}{k_B T} \qquad (4.19)$$

- We obtain ~~N~~ atomnumber

$$N = \frac{1}{(2\pi\hbar)^3}\int d^3\vec{r}\, d^3\vec{p}\; f(\vec{r}, \vec{p}) \qquad (4.20)$$

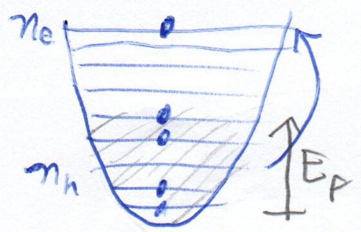
or density / momentum density

$$n(\vec{r}) = \frac{1}{(2\pi\hbar)^3}\int d^3\vec{p}\; f(\vec{r}, \vec{p})$$

$$\tilde{n}(\vec{p}) = \frac{1}{(2\pi\hbar)^3}\int d^3\vec{r}\; f(\vec{r}, \vec{p}) \qquad (4.21)$$

- The same view-point adapted here can give the T.F. profile for Bosons (see Eq (3.31)):
  In a (locally) homogeneous BEC there is no kinetic energy, but interaction energy $U_0 n(r)$. Replacing Fermi-(kinetic) energy by $U_0(n(r))$ in (4.17) then gives (3.31).

## 4.7.1. Excitations of the ideal gas




The simplest excited state of $|FS\rangle$ is when we move any atom with $E < E_F$ to $E > E_F$.

In this we are actually doing two things: creating a hole at $n_h$ and and excited atom at $n_e$:

(oscillator quantum number)

We can consider these seperately as excited states of the system with $N-1$ (hole) or $N+1$ (excited atom) atoms. Energy of hole: $E[\hat{a}_{n_h}|FS\rangle_N] - E[|FS\rangle_{N-1}]$

"energy of [...]"

$$= E_F - \underbrace{E_{n_h}}_{=\hbar\omega(n_h + \frac{1}{2})}$$

Similarly for excitation $E[\hat{a}^\dagger_{n_e}|FS\rangle_N] - E[|FS\rangle_{N+1}]$

$$= E_{n_e} - E_F$$

If we denote by $n_F$ the ocillator state until which all states are filled in the Fermi sea we have

$$\boxed{\text{Energy of particle \& hole excitations} \quad E_n = \hbar\omega|n - n_F| \quad (4.22)}$$

(homogeneous system would have $\varepsilon_k = \frac{\hbar^2|k^2 - k_F^2|}{2m}$)

# 4.8. (Weak) Repulsive interactions in spin-mixtures

- So far we considered only non-interacting atomic Fermi gases, which due to discussion in sec 4.6. is actually realistic for a cold single species gas
- For two species (e.g. $\frac{N}{2}$ atoms in $|\uparrow\rangle$, $\frac{N}{2}$ in $|\downarrow\rangle$) interactions become relevant since $|\uparrow\rangle$ atoms do have s-wave interactions with $|\downarrow\rangle$ atoms.
- thus also evaporative cooling works again

Let's assume fully repulsive interactions (True $U(r)>0$ $\forall r$)

## 4.8.1. Landau Fermi Liquid

Let us consider "slow" turning on of interactions, start with perturbation theory. Hamiltonian:

$$\hat{H} = \int d^3x \left\{ \sum_{s=\uparrow,\downarrow} \hat{\Psi}^\dagger_s(\vec{x})\, \hat{H}_0 \hat{\Psi}_s(\vec{x}) + U_0 \hat{\Psi}^\dagger_\uparrow(\vec{x}) \hat{\Psi}^\dagger_\downarrow(\vec{x}) \hat{\Psi}_\downarrow(\vec{x}) \hat{\Psi}_\uparrow(\vec{x}) \right\} \quad (4.23)$$

for Fermi-gas spin-mixture

- Field operator now has spin index

$$\hat{\Psi}_s(\vec{x}) = \sum_n \hat{a}_{s,n} \varphi_n(\vec{x}) \chi_s$$

($\chi_s$ = spinor, i.e. $s=\uparrow \to \begin{pmatrix}1\\0\end{pmatrix}$ $s=\downarrow \to \begin{pmatrix}0\\1\end{pmatrix}$)

($\hat{a}_{s,n}|0\rangle = |n, s\rangle$ $\quad s = |\uparrow\rangle, |\downarrow\rangle$)

$n$: ~~trap~~ single particle state

- We have $\{\hat{\Psi}_s(\vec{x}), \hat{\Psi}^\dagger_{s'}(\vec{x}')\} = \delta_{ss'}\, \delta^{(3)}(\vec{x}-\vec{x}')$ (4.24)

- Hamiltonian includes the fact that only atoms in two different spin-states can interact.

For simplicity consider homogeneous system

$$\boxed{\hat{\psi}_\sigma(\vec{x}) = \sum_{\vec{k}} \frac{\hat{a}_{\sigma,\vec{k}}}{\sqrt{(2\pi)^3}} \underbrace{\varphi_{\vec{k}}(\vec{x})}_{\text{plane-waves}} \chi_\sigma}$$

plane-waves, see p. 4

DO steps

Using $(2\pi)^3 \delta^{(3)}(\vec{k}) = \int d^3\vec{x}\, e^{i\vec{k}\vec{x}}$ we obtain

$$\hat{H} = \underbrace{\sum_k \frac{\hbar^2 k^2}{2m}\left(\hat{a}^+_{\uparrow k}\hat{a}_{\uparrow k} + \hat{a}^+_{\downarrow k}\hat{a}_{\downarrow k}\right)}_{\hat{H}_0} + \underbrace{\frac{U_0}{V}\sum_{\substack{k_1 k_2 \\ k_3 k_4}} \cancel{\delta^{(3)}(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)}}_{\text{(perturbation)}} \times$$

$$\sum_{\vec{k}_1+\vec{k}_2 = \vec{k}_3+\vec{k}_4}$$

$$\times\ \hat{a}^+_{\uparrow k_3}\hat{a}^+_{\downarrow k_4}\hat{a}_{\downarrow k_2}\hat{a}_{\uparrow k_1} \qquad \text{Momentum-space } \hat{H} \text{ spin-mixture} \qquad (4.24)$$

Find change of energy for small interactions $U_0$ using Rayleigh-Schrödinger perturbation theory

$$E^{(0)} = \langle FS|\sum_k \frac{\hbar^2 k^2}{2m}\left(\hat{a}^+_{\uparrow k}\hat{a}_{\uparrow k} + \hat{a}^+_{\downarrow k}\hat{a}_{\downarrow k}\right)|FS\rangle \qquad (\hat{H}_0 \text{ only!})$$

$$= \cancel{(4\pi)}\int_0^{k_{F,\uparrow}} dk\, k^2 \underbrace{D}_{\substack{= \left(\frac{L}{2\pi}\right)^3 \\ \text{density of states}}} \frac{\hbar^2 k^2}{2m} + \text{''}\ k_{F\uparrow} \to k_{F\downarrow}$$

density of states, see p. 54

$$\overset{(4.1),(4.3),D}{=\!=} \frac{3}{5}\left(E_{F,\uparrow} N_\uparrow + E_{F\downarrow} N_\downarrow\right) \qquad \begin{matrix}(4.25b) \\ \text{[c.f. } (4.4b))\end{matrix}$$

(We have incidentally derived, total energy of ideal Fermi gas:

$$E_{TOT} = \frac{3}{5} E_F N \qquad (4.25)$$

DRAW: Picture:
- need $k_1, k_2 < k_F$
- due to $\langle FS|k\uparrow,\downarrow$ $\to k_3 = k_1$, $k_2 = k_4$!

state factors in $\uparrow \otimes \downarrow$...

1st order energy correction

$$E^{(1)} = \langle FS|\hat{V}|FS\rangle \ \underset{k_2 = k_4}{\overset{\text{to get non-zero}\ k_1 = k_3}{=\!=\!=}}\ \frac{U_0}{V}\sum_{k_1 k_2}\langle \hat{a}^+_{\uparrow k_1}\hat{a}_{\uparrow k_1}\rangle\langle \hat{a}^+_{\downarrow k_2}\hat{a}_{\downarrow k_2}\rangle$$

$$= \frac{U_0}{V} N_\uparrow N_\downarrow \qquad \text{energy correction} \qquad (4.26)$$

(Shown here mainly as example for perturbation theory in a many-body context!)

Let us also look at the first order correction to the quantum state $|FS\rangle$: unperturbed state

From QM-I:
$$\underbrace{|n^{(1)}\rangle}_{\text{perturbed state}} = \sum_{k \neq n} \frac{\langle k^{(0)}|\hat{V}|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}} \underbrace{|k^{(0)}\rangle}_{\text{basis}}$$

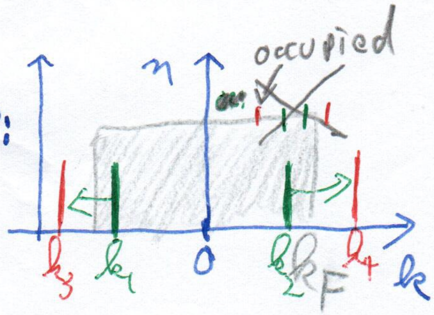
In our many-body context this translates to

$$|FS^{(1)}\rangle = \sum_{\vec{N}} \frac{\langle \vec{N}|\hat{V}|FS^{(0)}\rangle}{E^{(0)} - E_{\vec{N}}} |\vec{N}\rangle \qquad (4.27)$$

- Fock-states $|\vec{N}\rangle$, see (2.14), for fermions, taking into account spin also
- $E_{\vec{N}} = \sum_{k} \frac{\hbar^2 k^2}{2m} (N_{k\uparrow} + N_{k\downarrow})$
- $\hat{V}$, see (4.24)

Matrix-elements:
$$\langle \vec{N}|\hat{V}|FS^{(0)}\rangle = \frac{U_0}{V} \sum_{\substack{k_1,k_2,k_3,k_4 \\ k_1+k_2=k_3+k_4}} \underbrace{\langle \vec{N}| \hat{a}^\dagger_{\uparrow k_3} \hat{a}^\dagger_{\downarrow k_4} \underbrace{\hat{a}_{\downarrow k_2} \hat{a}_{\uparrow k_1} |FS^{(0)}\rangle}_{\substack{\text{we need} \\ |k_1|, |k_2| < k_F}}}_{\substack{\text{we need } k_3 = k_1 \,\&\, k_4 = k_2 \text{ OR (boring)} \\ |k_3|, |k_4| > k_F}}$$

e.g.:




$\Rightarrow$ We couple only to Fock-states that represent a filled Fermi sea, with <u>two particle-hole excitations</u>

We denote the latter <u>particle-hole</u> state with

$$|\underbrace{(k_3\uparrow)^e}_{\substack{\text{excitation with} \\ \text{wave vector } k_3, \\ \text{spin } \uparrow}} (k_4\downarrow)^e (k_2\downarrow)^h \underbrace{(k_1\uparrow)^h}_{\substack{\text{hole with} \\ \text{wave vector } \vec{k}_1 \\ \text{spin } \uparrow}}\rangle \qquad (4.28)$$

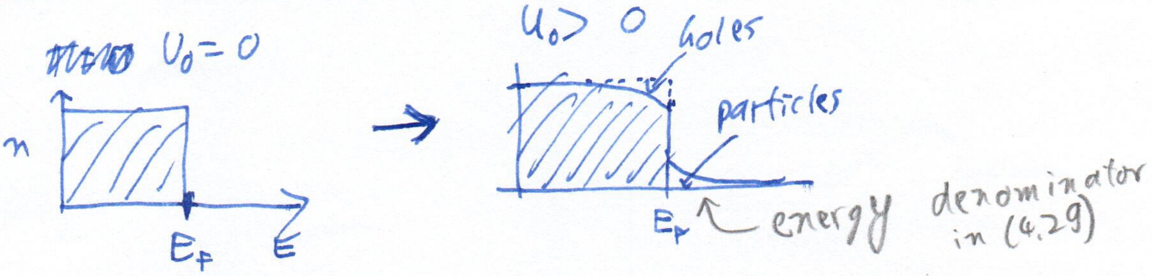
e.g.

The perturbed Fermi-sea from (4.27) thus is [$E^{(0)}$ fr. (4.25b)]

$$|FS'\rangle + |FS^{(0)}\rangle + \frac{U_0}{V}\sum_{k_1+k_2=k_3+k_4} \frac{|(k_3\uparrow)^e (k_4\downarrow)^e (k_2\downarrow)^h (k_1\downarrow)^h\rangle}{E^{(0)} - \underbrace{\left[\sum_{l=1..4} \frac{\hbar^2 |k_l^2 - k_F^2|}{2m} + E^{(0)}\right]}_{(4.22)\text{ was relative to } E^{(0)}}} \quad (4.29)$$

It is said that the ~~int~~ interactions dress the FS with particle + hole pairs.

this leads to a softening of the Fermi edge even at $T=0$

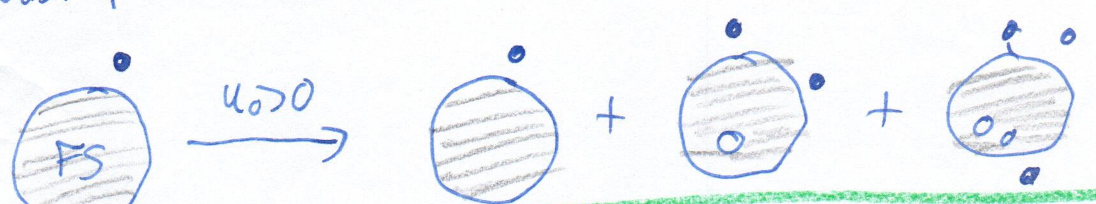



Also excited-states get dressed with other many-body states.

Fermi-liquid theory can be understood as free fermions $|k, \sigma\rangle$ evolving into fermionic quasi-particles with ~~spin~~ the same ~~momentum~~ and spin, due to interactions / dressing. These have a slightly modified effective mass $m^*$.

~~For that reason~~

Quasi particle cartoon




define again → p. 72b

- Most properties of Fermi-liquid system are (surprisingly) similar to the non-interacting cases
- Describes most non-superconducting metals.
- cold-atom experiments:
  See Nascimbène et al. Nature 463 1057 (2010)
  Horikoshi et al. Science 327 442 (2010)