Week 4

PHY 635 Many-body Quantum Mechanics of Degenerate Gases
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2.4 Coherent states

Coherent states are a very useful concept in many areas of quantum physics. We discuss two types, which are mathematically/algebraically identical but conceptually subtly different:

2.4.1 Coherent Harmonic Oscillator States

- Question: What is the "most classical" type of oscillation we can get in the quantum harmonic oscillator:
- Answer: Define

Coherent State

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \exp[\alpha \hat{b}^\dagger]|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle, \ \ \alpha \in \mathbb{C} \tag{2.42}$$

We can write this also as: $|\alpha\rangle = \mathrm{D}(\alpha)|0\rangle$, where $\mathrm{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$ is the displacement operator.

- Here $\hat{b}^\dagger$ is a ladder operator from (1.12).
- Coherent states are not necessarily eigenstates of the harmonic oscillator Hamiltonian $\hat{H}_{SHO}$ in (1.13), since they have an uncertain energy/number of oscillator quanta.

Properties of coherent states

$$\hat{b}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \hat{b}^\dagger|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle, \qquad \langle\alpha|\hat{b}^\dagger = \langle\alpha|\alpha^*, \tag{2.43}$$

$$\langle\alpha|\alpha'\rangle = \exp\left[\alpha^*\alpha' - \frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2}\right] \ (\underline{\text{Not orthogonal}}), \tag{2.44}$$

$$\langle\alpha|\alpha\rangle = 1, \tag{2.45}$$

$$\mathbb{I} = \frac{1}{\pi}\int d\alpha|\alpha\rangle\langle\alpha|. \tag{2.46}$$

- Coherent state is a right-eigenstate of destruction operator
- Two different coherent states are typically not orthogonal, unless $|\alpha - \alpha'|$ is very large (and even then only approxiately).

Proof of (2.43): Let $|\bar{\alpha}\rangle = e^{\frac{|\alpha|^2}{2}}|\alpha\rangle$

$$\begin{aligned}
\hat{b}|\bar{\alpha}\rangle &= \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{b}|\varphi_n\rangle = \sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|\varphi_{n-1}\rangle \\
&\underset{n\mapsto n+1}{=} \sum_{n=0}^{\infty}\frac{\alpha^{n+1}}{\sqrt{(n+1)!}}\sqrt{n+1}|\varphi_n\rangle = \alpha\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\varphi_n\rangle = \alpha|\bar{\alpha}\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{b}^\dagger|\bar{\alpha}\rangle &= \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{b}^\dagger|\varphi_n\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n+1}|\varphi_{n+1}\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\frac{1}{n+1}\frac{\partial}{\partial\alpha}\alpha^{n+1}\sqrt{n+1}|\varphi_{n+1}\rangle = \sum_{n=0}^{\infty}\frac{\partial}{\partial\alpha}\frac{\alpha^{n+1}}{\sqrt{(n+1)!}}|\varphi_{n+1}\rangle \\
&= \frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\varphi_n\rangle = \frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\varphi_n\rangle = \frac{\partial}{\partial\alpha}|\bar{\alpha}\rangle
\end{aligned}$$

(Rest follows from the product rule.)

Example: Oscillation of coherent state: What is the meaning of α? Let $\alpha_0 \in \mathbb{R}$ Convert the equation $\hat{b}|\alpha_0\rangle = \alpha_0|\alpha_0\rangle$ to the position basis:

$$\langle x|\hat{b}|\alpha_0\rangle = \alpha_0 \underbrace{\langle x|\alpha_0\rangle}_{\equiv\tilde{\alpha}_0(x)}$$

$$\xrightarrow[(1.4)]{\text{Using}} \quad \frac{\partial}{\partial x}\tilde{\alpha}_0(x) = \bigg(-\frac{m\omega}{\hbar}x + \sqrt{\frac{2m\omega}{\hbar}}\alpha_0 \bigg)\tilde{\alpha}_0(x)$$

$$\xrightarrow[\text{DE}]{\text{Solve}} \quad \tilde{\alpha}_0(x) = C\ \exp\bigg[-\frac{(x-\alpha_0')^2}{2\sigma^2} \bigg],$$

where $\sigma = \sqrt{\frac{\hbar}{m\omega}}$ and $\alpha_0' = \sqrt{2}\sigma\alpha_0$. Thus the position space representation of a coherent state has a Gaussian shape, with center location governed by α_0'.
We now want to find the time evolution of the coherent state $|\,\alpha_0\,\rangle$. The latter is assembled from oscillator eigen states that obey:

$$\hat{H}_0|\varphi_n\rangle = E_n|\varphi_n\rangle, \quad E_n = \hbar\omega\bigg(n + \frac{1}{2}\bigg).$$

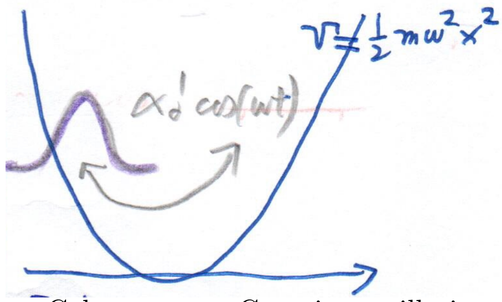
Since the Hamiltonian is time-independent, we can us the standard rules for time evolution to find

$$\begin{aligned} \Rightarrow |\alpha(t)\rangle &= \sum_n \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\omega(n+\frac{1}{2})t}|\varphi_n\rangle \\ &= \sum_n \frac{1}{\sqrt{n!}}(\alpha_0 e^{-i\omega t})^n e^{-i\frac{\omega}{2}t}|\varphi_n\rangle \\ &= e^{-i\frac{\omega}{2}t}|\alpha_0 e^{-i\omega t}\rangle. \end{aligned}$$

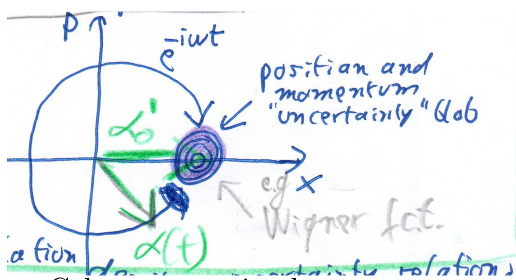
Can show after some fiddling:

$$|\tilde{\alpha}(x,t)|^2 = C'\exp\bigg[-\frac{(x-\alpha_0'\cos(\omega t))^2}{\sigma^2} \bigg]$$

.
We thus always have a ground-state shaped Gaussian oscillating in the potential with amplitude α_0'.



top: Coherent state Gaussian oscillating in a harmonic trap



top: Coherent state in phase space, represented by Wigner function (see below)

2.4.2 Wigner function

In the example above, bottom right, we also wanted to show a phase space representation of a quantum harmonic oscillator in a coherent state.

Classically we have the idea of phase-space (x, p). Quantum mechanically $\Delta x \Delta p \geq \hbar/2 \rightarrow$ particle cannot have a fixed phase-space coordinate. We can still represent a quantum state $\varphi(x)$ in phase-space, using the

Wigner distribution

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \varphi^*(x+y)\varphi(x-y)e^{2ipy/\hbar}dy \tag{2.47}$$

- Properties

$$\int_{-\infty}^{\infty} dp\, W(x,p) = |\varphi(x)|^2 \text{ (position-space distribution)},$$

$$\int_{-\infty}^{\infty} dx\, W(x,p) = |\tilde{\varphi}(p)|^2 \text{ (momentum-space distribution)}.$$

- $W(x, p)$ is a quasi-probability distribution (means we can get some expectation values by integrating over it, but it may have regions with $W(x, p) < 0$)
- The interpretation is that when drawing $W(x, p)$, non-zero regions show the location of a quantum-state in phase-space. This was used in the figure of the example above.

We can alternatively define the

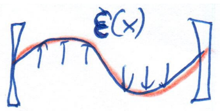
Wigner function from the number-state representation

$$\chi_W(\lambda, \lambda^*) = Tr\{\hat{\rho}e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}\} \tag{2.48}$$

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2}\int d^2\lambda e^{-\lambda\alpha^* + \lambda^*\alpha}\chi_W(\lambda, \lambda^*) \tag{2.49}$$

- The above gives the same as (2.47) for harmonic oscillator ladder operators $\hat{a} \rightarrow \hat{b}$.
- It directly generalizes to Fock states (2.2), when $\hat{a}$ are many-body creation and destruction opertors.

Example, Laser:

Consider a single-mode photon field at frequency ω: 

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}, \text{ just as for oscillator}$$

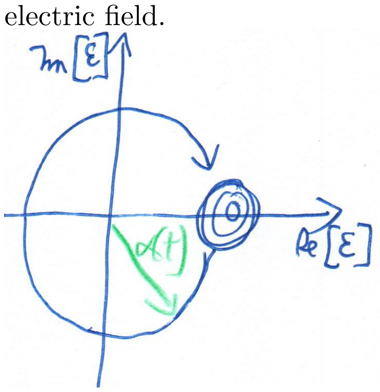
Electric field (c.f. Example C page 19)

$$\hat{E}(x,t) = \mathcal{E}(x,t)\hat{a} + h.c.$$

Taking expectation value in the coherent state $|\alpha(t)\rangle$, we can show (exercise)

$$\langle \alpha(t)|\hat{E}(x,t)|\alpha(t)\rangle = 2\,\mathfrak{Re}\{\mathcal{E}(x,t)\underbrace{\alpha_0 e^{-i\omega t}}_{\alpha(t)}\}$$

Thus here, the complex number $\alpha(t)$ characterizes amplitude and phase of the oscillating electric field.



2.4.3 Coherent many-body states

Due to identical properties of ladder $\hat{b}$ operators and $\hat{a}, \hat{c}$, we can equally define a

Many-body coherent state (Bosons):

$$|\alpha\rangle = \exp[\hat{a}_m^\dagger \alpha]|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \alpha \in \mathbb{C} \tag{2.50}$$

where $|n\rangle$ is a Fock-state that represents the occupation of mode $|\phi_m\rangle$.

- this now describes a superposition of different occupation numbers (Fock-states) of single-body mode $|\varphi_m\rangle$

- all properties of (2.43)-(2.46) apply

We can combine states (2.50) for multiple single-particles states (modes) into

Many-mode coherent state (Bosons):

$$\hat{a}_k|\boldsymbol{\alpha}\rangle = \alpha_k|\boldsymbol{\alpha}\rangle, \quad \boldsymbol{\alpha} = \{\alpha_1 ... \alpha_N\}, \quad \alpha_k \in \mathbb{C} \tag{2.51}$$

which exhibit one coherent amplitude α_k for each single-particle basis state k

- The slightly messy formal decomposition of (2.51) into Fock-states is

$$|\boldsymbol{\alpha}\rangle = e^{-\sum_k \frac{|\alpha_k|^2}{2}} \sum_{n_1 n_2 ... n_N} \frac{\alpha_1^{n_1} \alpha_2^{n_2} ... \alpha_N^{n_N}}{\sqrt{n_1!}\sqrt{n_2!}...\sqrt{n_N!}} |n_1 n_2 ... n_N\rangle. \tag{2.52}$$

2.4.4 Fermionic coherent states (not used here)

If we assume a definition like (2.51) for fermionic operators we run into trouble:

$$\{\hat{a}_k, \hat{a}_l\}|\boldsymbol{\alpha}\rangle = (\alpha_k\alpha_l + \alpha_l\alpha_k)|\boldsymbol{\alpha}\rangle \stackrel{!}{=} 0 \qquad (\text{since}\{\hat{a}_k, \hat{a}_l\} = 0)$$

For two non-zero complex numbers $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\alpha_k\alpha_l \neq 0$ of course.

Solution: We use

Grassmann-numbers Defined as an anti-commuting set of complex numbers

- Based on this we can also use the coherent state concept for fermions. Mainly useful for fermionic path integrals
- Not further used in this lecture