PHY635, I-Semester 2019/20, Assignment 5, solution

Instructor: Sebastian Wüster

(1) Condensate depletion Consider a homogenous 3D BEC, i.e. $|\phi_0(\mathbf{x})|^2 = \rho_0 = const \neq 0$ and $V(\mathbf{x}) = 0$.

(i) Calculate the relative density of non-condensed atoms versus condensed ones at zero temperature $T = 0$ but non-zero interactions. These uncondensed atoms are called depletion. [*Hint: Convert discrete sums that occur into integrations.*][*5 points*].

(ii) Interpret the result in terms of physics. [*3 points*]

(iii) How can you use it to assess the validity of Bogoliubov theory? [*2 points*]

Solution:

(i) The density of non-condensate atoms at $T = 0$ is given by the third term of Eq 3.38 of the lecture notes as,

$$\begin{aligned} n_{ex} &= \frac{1}{V} \sum_{\mathbf{p}(\mathbf{p}\neq\mathbf{0})} v_p^2. \\ &= \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} v_p^2. \end{aligned} \tag{1}$$

Using the expression of v_p^2 from Eq 3.71 of the lecture notes, the integration in the spherical polar cordinate is given as:

$$n_{ex} = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{1}{2} \left(\frac{\xi_p}{\epsilon_p} - 1 \right), \tag{2}$$

where $\xi_p = \epsilon_0^p + n_0 U_0$ and $\epsilon_p = \sqrt{(\epsilon_0^p)^2 + 2n_0 U_0 \epsilon_0^p}$, see lecture notes.

Using Mathematica the value of the integration is given as:

$$n_{ex} = \frac{1}{3\pi^2} \left(\frac{mc}{\hbar} \right)^3, \tag{3}$$

where $c = \sqrt{n_0 U_0 / m}$ is the speed of sound.

The relative density of the uncondensed atoms is thus:

$$\frac{n_{ex}}{n_0} = \frac{8}{3\sqrt{\pi}} (n_0 a_s^3)^{1/2}, \tag{4}$$

where a_s is the scattering length.

(ii) *Physically this result may be understood by noting that v_p^2 is order of unity for momenta $p \approx \hbar/\xi$, and falls off rapidly at larger momenta. The number density of particles in the excited states is thus of order the number of states per unit volume with wave number less than $1/\xi$, that is, $1/\xi^3$ in three dimensions.*

(iii) *If d is the particle spacing in the condensate, then the Eq. (4) can also be expressed as:*

$$\frac{n_{ex}}{n_0} \sim \frac{8}{3\sqrt{\pi}}\left(\frac{a_s}{d}\right)^{3/2}. \tag{5}$$

For Bogoliubov theory to be valid, we require that $n_{ex} \ll n_0$, i.e. the uncondensed density is much less than the condensed one. This is in fact the same statement as having a dilute condensate $d \ll a$.

(2) Bogoliubov excitations in a harmonic trap *Consider a BEC in the Thomas-Fermi approximation in a quasi-1D harmonic trap, with trapping frequency ω_x and interaction strength U_{1d}. Solve the BdG equations (3.61), for very energetic modes. Take "very energetic" to imply a mode energy $E_n \gg \mu$ and $E_n \gg U_{1d}|\phi_0(0)|^2$. [Hint: You may use known solutions of the TISE for the harmonic oscillator.] [10 points] Note: Skip $E_n \gg \hbar\omega_x$ from the question sheet, this might have been confusing.*

Solution

(i) *The Bogoliubov-de-Gennes (BdG) equations in 1D:*

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_x x^2 + 2U_{1D}|\phi_0(x)|^2 - \mu - E_n\right]u_n(x) = U_{1D}\phi_0(x)^2 v_n(x)$$
$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_x x^2 + 2U_{1D}|\phi_0(x)|^2 - \mu + E_n\right]v_n(x) = U_{1D}(\phi_0(x)^*)^2 u_n(x) \tag{6}$$

Applying the condition given in the question for very energetic modes i.e. $E_n \gg \mu$ and $E_n \gg U_{1D}|\phi_0(x)|^2$, the above equations can be expressed as:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u_n(x)}{\partial x^2} + \frac{1}{2}m\omega_x x^2 u_n(x) = E_n u_n(x)$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2 v_n(x)}{\partial x^2} + \frac{1}{2}m\omega_x x^2 v_n(x) = -E_n v_n(x) \tag{7}$$

We see that these mostly have a similar structure as the SE for the harmonic oscillator, so let's make the Ansatz $u_n(x) = \bar{u}\varphi_n(x)$, $v_n(x) = \bar{v}\varphi_n(x)$, where $\varphi_n(x)$ is the n'th harmonic oscillator solution.[1] *Using the properties of SHO solutions, we then have*

$$(\hbar\omega_n - E_n)\bar{u} = 0$$
$$(\hbar\omega_n + E_n)\bar{v} = 0. \tag{8}$$

[1] Note that while the equations for u and v have decoupled, we still have to consider the solutions jointly due to the $\bar{u}^2 - \bar{v}^2 = 1$ normalisation condition.

We have to solve this together with the normalisation condition $|\bar{u}|^2 - |\bar{v}|^2 = 1$. The only solution for both is $\bar{u} = 1$, $\bar{v} = 0$, $E_n = \hbar\omega_n$.

(3) White dwarfs *Show that the radius of a white dwarf is given by Eq. (4.12). [10 points]*

Solution *In a stable star, the differential change of energy dE, for a change of radius dR should be zero. This is expressed in Eq. (4.11) of the notes.*

$$0 = dE = \frac{\partial}{\partial R}\Big(\underbrace{-\frac{3}{5}\frac{M^2}{R}G}_{E_{grav}}\Big)dR - P_F(R)(4\pi R^2 dR) \tag{9}$$

The first part uses the total gravitational energy of a uniform sphere with mass M as $E_{grav} = -\frac{3}{5}\frac{M^2}{R}G$ and the last part uses the thermodynamic relation $dE = -PdV$. Of course approximating the star as having uniform density is only an approximation.

The Fermi pressure can be given by the Eq 4.10 of the lecture notes:

$$\begin{aligned}
P_F(R) &= \frac{2}{5}\left(\frac{N}{V}\right)E_F \\
&= \frac{2}{5}\left(\frac{N}{V}\right)(3\pi^2)^{2/3}\frac{\hbar^2}{2m_e}\left(\frac{N}{V}\right)^{2/3} \\
&= \frac{2}{5}(3\pi^2)^{2/3}\frac{\hbar^2}{2m}\rho_e^{5/3}, \rho_e = (N/V)
\end{aligned} \tag{10}$$

Here ρ_e is the number density of electrons. To get this we first calculate the mass density in the star $\rho = M/(\frac{4}{3}\pi R^3)$, get the number density of atoms as ρ/m_{He}, where m_{He} is the mass of a Helium atom, and then multiply by two for the two electrons inside. Hence:

$$\rho_e = \frac{M}{\frac{4}{3}\pi R^3}\frac{2}{m_{He}}. \tag{11}$$

Inserting ρ_e into (10) and solving the equation for R, we obtain result (4.12) of the lecture.

(4) Bosonic versus Fermionic ground-states

The template file `Assignment5_phy635_program_draft_v1.xmds` *finds the ground-state of the Schrödinger equation for two Bosonic ^7Li atoms in a one-dimensional harmonic trap using imaginary time evolution. Lithium also has a long-lived Fermionic isotope ^6Li.*

(4a) From the many-body wavefunction, derive an expression for the total density of atoms at position x. Implement the sampling of that in the last output block of the script provided. Note that the block is set up to integrate whatever is inserted over the coordinate x_2. [2 points].

(4b) Show analytically that the imaginary time (and real time) Schrödinger equation for two identical *particles preserves Bosonic and Fermionic symmetries of the wave-function. [3 points]*

(4c) Using (4b), modify the code such that it can find the corresponding ground-state for two Fermionic atoms. Compare total densities for the Fermionic and Bosonic cases with the scripts provided. How is the Fermionic density pattern called? [5 points]

Solution

(4a) If $\psi(x_1, x_2)$ is the manybody wavefunction of the 2 Li atoms, then total density of the atoms at some point x_1 is given as:

$$\rho(x) = 2 \int dx_2 |\psi(x, x_2)|^2 \tag{12}$$

In general we would have that the total density is $\rho(x) = \sum_n \rho(x_n)$, where $\rho(x_n)$ is the density for atom number n. We get the density for a single atom, by integrating over all other coordinates in the many-body wave function: $\rho(x_k) = \int dx_1 dx_2 \dots \text{skip } dx_k \dots dx_N |\psi(x_1, x_2, \dots, x_k, \dots, x_N)|^2$. Since the many-body wave function fulfills Bose symmetry all N pieces in $\rho(x) = \sum_n \rho(x_n)$ will be identical. Hence we can just consider the density for atom one only and have $\rho(x) = \int dx_2 \dots dx_N |\psi(x, x_2, \dots, x_k, \dots, x_N)|^2$.

(4b) We do this for two particles only (more follows similarly): Let us assume our wavefunction fulfills $\psi(x_1, x_2) = \pm\psi(x_2, x_1)$. Identical particles must feel the same potential $V(x)$ and have the same mass m, hence the SE is

$$i\hbar \frac{\partial}{\partial t} \psi(x_1, x_2) = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + V(x_1) + V(x_2) \right] \psi(x_1, x_2). \tag{13}$$

We can directly see by inspection of the right-hand side that $i\hbar \frac{\partial}{\partial t} \psi(x_1, x_2) = \pm i\hbar \frac{\partial}{\partial t} \psi(x_2, x_1)$. Hence if the wave-function fulfills Bose or Fermi symmetry initially, it will do so at all later times as well.

(4c) Fig. 1 is the ground state of the two-body wavefunction for the Fermions and Bosons in the harmonic trap. It is clear from the figure that time imaginary evolution preserve the Bosonic and Fermionic symmetry of the manybody wavefunction.

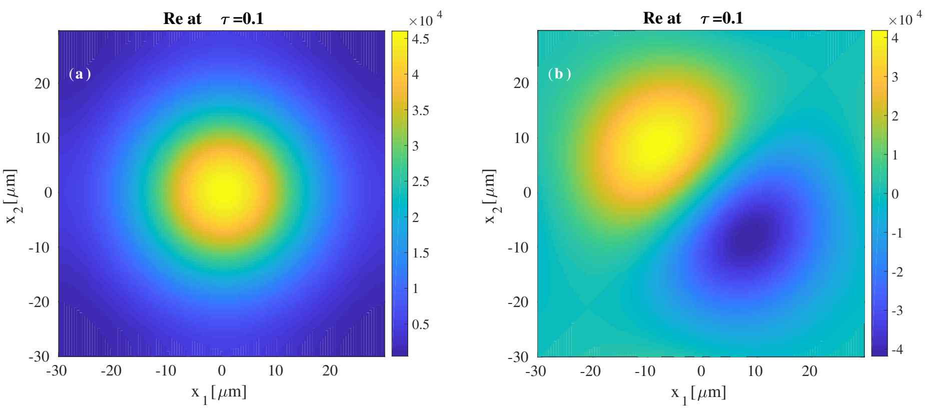

Fig. 1. Panel (a) and panel (b) are the real parts of the wavefunction for Two Bosons and Fermions in the harmonic trap.

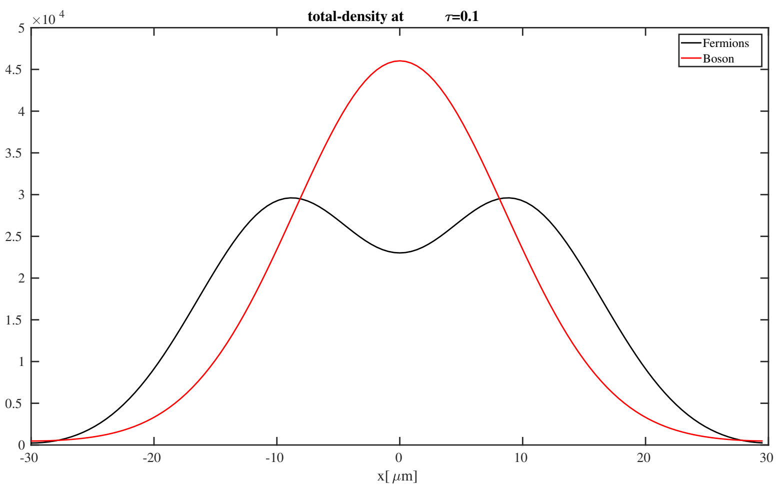

Fig. 2. Comparision of the total density for two Bosons and Fermions in the harmonic trap.