

# PHY635, I-Semester 2019/20, Assignment 4 solution

Instructor: Sebastian Wüster

(1) **Gross-Pitaevskii equation** Consider some special solutions of the TIGPE (3.44). Let us assume a 1D system without any external trap  $V(x) = 0$ .

- (i) For the case of attractive interactions,  $U_0 < 0$ , show that one solution for  $N$  atoms at a chemical potential  $\mu$  is given by

$$\phi_0(x) = A \operatorname{sech}(x/\xi), \quad (1)$$

This is called a bright soliton. While showing it is a solution, find the values for  $A$  and  $\xi$  required. [5 points]

*Solution: Let us insert (1) into the TIGPE and evaluate the rhs and lhs. We find*

$$\mu A \operatorname{sech}(x/\xi) = -\frac{A\hbar^2}{2m\xi^2} \left( -\operatorname{sech}^3(x/\xi) + \operatorname{sech}(x/\xi) \tanh^2(x/\xi) \right) - |U_0| A^3 \operatorname{sech}^3(x/\xi). \quad (2)$$

*Dividing both sides by  $A \operatorname{sech}(x/\xi)$  we have*

$$\mu = -\frac{\hbar^2}{2m\xi^2} \left( -\operatorname{sech}^2(x/\xi) + \tanh^2(x/\xi) \right) - |U_0| A^2 \operatorname{sech}^2(x/\xi). \quad (3)$$

*Rewriting the rhs:*

$$\mu = -\frac{\hbar^2}{2m\xi^2} + \frac{(\hbar^2 - Am|U_0|\xi^2)}{m\xi^2} \operatorname{sech}^2(x/\xi). \quad (4)$$

*Since the lhs is constant, we need  $(\hbar^2 - Am|U_0|\xi^2) = 0$ . Afterwards  $\mu = -\frac{\hbar^2}{2m\xi^2}$ . Finally, from normalisation  $\int_{-\infty}^{\infty} dx |A|^2 \operatorname{sech}^2(x/\xi) = 2\xi |A|^2 = N$ . The last three equations together have the solution:*

$$A = \frac{\sqrt{mN} \sqrt{|U_0|}}{2\hbar}, \quad (5)$$

$$\xi = \frac{2\hbar^2}{mN|U_0|}, \quad (6)$$

$$\mu = -\frac{mN^2|U_0|^2}{8\hbar^2}. \quad (7)$$

- (ii) Plot the atom density for some realistic parameters. Start with the mass of Lithium and  $a_s = -1a_0$  (where  $a_0$  is the Bohr radius) but then use  $U_0 = U_{1d} = \frac{4\pi\hbar^2 a_s}{m} / (2\pi\sigma_{\perp}^2)$  with  $\sigma_{\perp} = (2\pi)10000$  Hz. We have to use  $U_{1d}$  because of our one dimensional calculations. Also use  $N = 5000$ . Then plot it for some varied values of  $N$  and  $a_s < 0$ . [5 points] *Solution: See Fig. 1.*

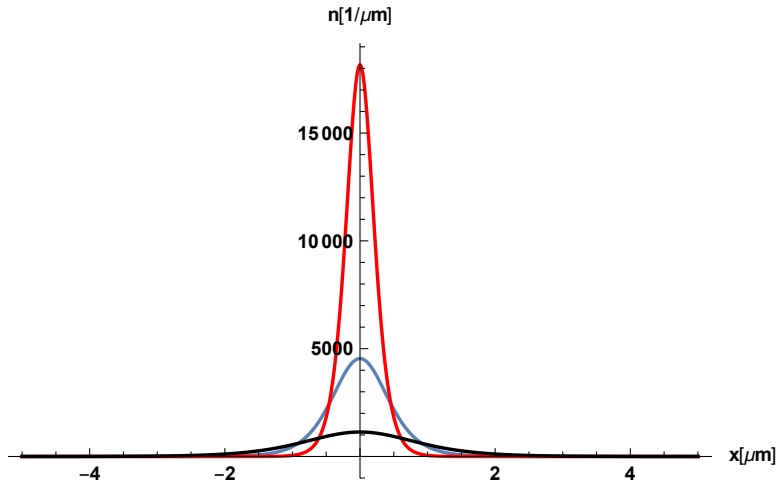


Abbildung 1: Three solitons for the  $a_s$  from the text and  $N = 2500$  (black),  $N = 5000$  (blue)  $N = 10000$  (red).

- (iii) Google the term soliton and explain in your own words what it implies. [5 points]  
*Solution: A soliton is a special solution to a non-linear wave-equation. It can propagate without change of shape, since dispersion is exactly balanced by the non-linearity. Two solitons also emerge from collisions as if they passed through each other. They are extensively used for communication in non-linear optical fibres.*

**(2) Bogoliubov excitations and stability** Consider the Bogoliubov dispersion relation given in the lecture.

- (i) Suppose we have a homogenous condensate with attractive interactions  $U_0 < 0$ . Discuss the BdG energies, depending on the excitation wave-number  $q$ . How does the time-evolution of the modes look, according to Eq. (3.73) of the lecture. Identify stable and unstable modes. [5 points].

*Solution: BdG energies are  $\hbar\omega_q = \sqrt{E_q(E_q + 2U_0\rho)}$  with  $E_q = \hbar^2q^2/2m$ . For  $U_0 < 0$  these energies become complex for all the modes with  $E_q < |2U_0\rho|$  (large wavelength modes). If a frequency has obtained an imaginary part (let  $\omega_q = \omega'_q + i\omega''_q$ , then in Eq. (3.73) the imaginary part will cause a rising exponential  $\hat{\psi}(x, t) \sim e^{|\omega''_q|t}$ . This indicates instability through exponential growth of the perturbation. Thus all modes with  $E_q < |2U_0\rho|$  are unstable.*

- (ii) Suppose you have a finite size system, such as a box of size  $L$ . How would this change the stability picture? [5 points]

*Solution: In that case, there is a smallest  $q = q_{min}$  or largest wave-length  $\lambda_{max}$  that any mode can have: When  $\lambda_{max} = L/2$  (think of quantum particle in a box). Thus  $q_{min} = \pi/L$ . If  $\hbar^2q_{min}^2/2m > |2U_0\rho|$ , all modes are stable. The system can thus exist despite attractive interactions. The BEC is however only metastable, since BdG modes only explore what happens to small perturbations.*

- (iii) Explain in your own words why excitations with wavelengths larger than the healing length are referred to as “collective excitations” (or sound waves) and those with shorter wavelengths as “single particle excitations”. [5 points]

*Solution: Don't know what your own words are. See lecture notes for my words.*

### (3) Numerical Solution of Gross-Pitaevskii equation

The template file `Assignment4_phy635_program_draft_v1.xmids` is set up to first find a ground-state of the TIGPE using “imaginary time evolution”, and then evolve that state in time. The potential imposes a hard-walled box, as discussed in section 3.3.5 of the lecture. During the ground-state finding the codes uses an interaction strength  $U_{\text{ini}}$  the time evolution uses  $U_{\text{fin}}$ .

- (3a) Verify the ground-state looks like the figure in section 3.3.5. Checkout the size of the edge transition region for various values of  $U_{\text{ini}}$  to confirm Eq. (3.50). Also check that the found ground-state does not evolve in time. [5 points] *Solution: See Fig. 2 for the overall box solution. See Fig. 3 for the atom density at the edge for three different interaction strength. The size of the drop from bulk-density to zero should roughly match the healing length  $\xi$ . For the parameters in Fig. 3 we find  $\xi = 0.3\mu\text{m}$  (black)  $\xi = 0.6\mu\text{m}$  (blue) and  $\xi = 0.15\mu\text{m}$  (red). The region itself is slightly larger, but the scaling with  $U$  is as expected.*

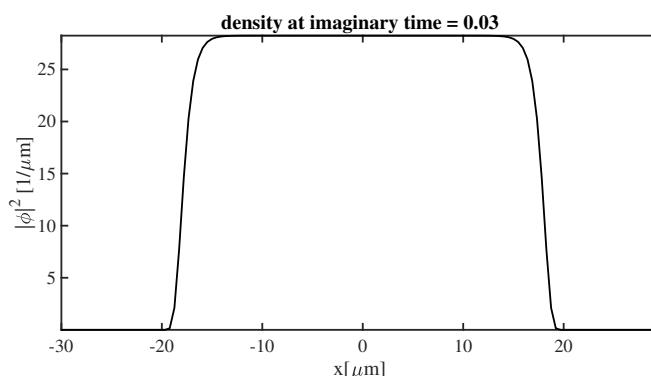


Abbildung 2: Atom density in a hard walled box potential from imaginary time evolution.

- (3b) Now change the interaction strength  $U_{\text{fin}}$  used during time-evolution to an attractive value ( $< 0$ ). Describe what you see and make some plots. Discuss wrt. your results of question 2. [5 points]

*Solution: We see initially the development of lots of spikes. These are the unstable modes discussed in Q2. Later evolution becomes messy. If the simulation would provide an energy relaxation channel (which it does not), the system would actually relax into the soliton waveforms discussed on Q1, which are the new stable ground-state at  $U_0 < 0$ .*

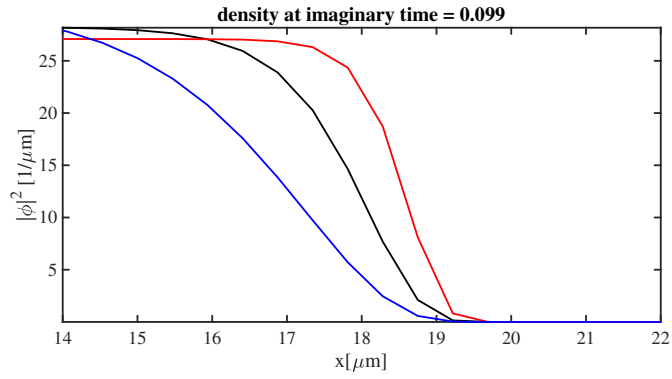


Abbildung 3: Zoom onto the edge of Fig. 2 for three different values of  $U_0$ : as in script provided (black), 4 times smaller (blue) 4 times larger (red).

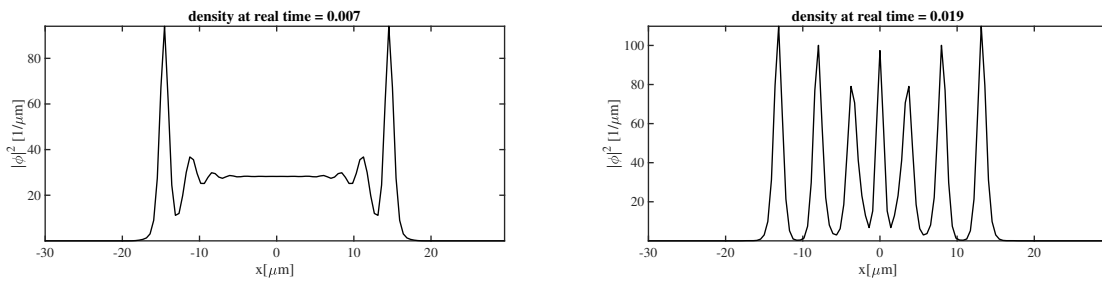


Abbildung 4: Dynamical instability of an almost homogenous BEC at two different snapshots after changing  $U_{\text{ini}} > 0$  to  $U_{\text{fin}} < 0$ .