# PHY635, I-Semester 2019/20, Assignment 3

*Instructor: Sebastian Wüster*
*Due-date: TA-Class, 13.9.2019*

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### (1) Gross-Pitaevskii equation

Consider $N$ Bosonic atoms of mass $m$ in a 3D isotropic harmonic trap $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$ that undergo Bose-Einstein condensation.

(i) Discuss in your own words why we can use an approximate delta-function contact interaction to deal with scattering of the ultra-cold, condensed atoms, and why that is useful. [*4 points*]

(ii) Now use a field-operator $\hat{\Psi}(\mathbf{x})$ to describe these atoms, take into account contact interactions as discussed in the lecture and derive an equation of motion for the field operator $\hat{\Psi}(\mathbf{x})$. [*4 points*]

(iii) Now assume the a field-operator acquires a non-vanishing expectation value upon condensation, such that $\langle\hat{\Psi}\rangle \approx \phi_0$, and find an equation for $\phi_0(\mathbf{x}, t)$. *You may approximate* $\langle\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}\rangle \approx \phi_0^*\phi_0\phi_0$. Describe the physical meaning of each piece in the Hamiltonian. [*3 points*]

(iv) Can you also reach a simple equation such as the above without the factorization assumption $\langle\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}\rangle \approx \phi_0^*\phi_0\phi_0$? If you try to avoid that assumptions, how could you try to proceed to find equation(s) anyway? [*4 points*]

**(2) Bose-Einstein condensation in varying dimensions** Reconsider the derivation of Bose-Einstein condensation in section 3.2. of the lecture notes. Instead of the special example of a 3D isotropic trap, assume a generic scenario where you assume a density of states $g(E) = c_\alpha E^{\alpha-1}$.

(i) Derive the critical temperature in terms of this density of states [*5 points*]. *Hint: In the lecture we are at some point summing over all states. Note the expression only depends on the energy of these states. Assume a dense continuum of states and convert that sum into an integration, using $g(E)$. Google "density of states" if needed.*

(ii) What is the density of states in a 1D harmonic oscillator potential? [*5 points*]

(iii) Based on your results of (i) and (ii), contemplate Bose-Einstein condensation in a strictly 1D harmonic oscillator potential? [*5 points*]

**(3) Numerical Solution of Gross-Pitaevskii equation**

The attached template file `Assignment3_phy635_program_draft_v1.xmds` is set up to first find a ground-state of the Schrödinger equation with a method called "imaginary time evolution"(see towards end of week 6), and then evolve that state in time. The ground-state finding uses a harmonic trap with frequency $\omega_{\text{ini}}$ the time evolution uses $\omega_{\text{fin}}$.

(4a) First test with `density_slideshow_v1.m` that for $\omega_{\text{ini}} = \omega_{\text{fin}}$ the "imaginary time evolution" converges to the ground-state despite the silly initial state and the state found later does not change in real time [2 points]. *Hint: note the first 101 time-samples are from the imaginary time evolution, the last 100 from the real time, the script knows this.*

(4b) Now slightly change $\omega_{\text{fin}}$ such that $\omega_{\text{fin}} \neq \omega_{\text{ini}}$. What happens? Use `plot_widths_v1.m` to plot the time evolution of the position uncertainty $\Delta\hat{x}$. Quantify what you see and compare with the final harmonic trap frequency. [2 points]

(4c) Change everything in the script that needs changing in order to tackle the Gross-Pitaevskii equation with the same traps, but 1000 atoms instead of one. Instead of $U_0$ from equation (4.8) we shall use an effective 1D interaction strength $U_{1d}$ that is already defined in the code. [2 points]

(4d) Redo the same as steps (a) and (b) for the Bose-Einstein condensate and discuss your findings. Use a couple of different $\omega_{\text{fin}}$ and try to deduce a law for what you see. [4 points]