PHY635, I-Semester 2019/20, Assignment 2

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Due-date: TA-Class, 30.8.2019

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(1) Ideal Bose gas, density fluctuations: [In this question, supply the details for the (newly updated) example in section 2.3.1]. Consider N Bosonic atoms in a 1D harmonic trap. To measure local density, we count atoms in a small region of size L, which corresponds to the operator

$$\hat{n}_{\text{loc}}(x_0) = \int_{x_0}^{x_0+L} dx \;\; \hat{\Psi}^{\dagger}(x)\hat{\Psi}(x), \tag{1}$$

and then use $\hat{\rho} = n_{\text{loc}}(x_0)/L$ to get a density.

Let us define the local number uncertainty

$$\Delta n_{\text{loc}}(x_0)^2 = \langle \hat{n}_{\text{loc}}(x_0)^2 \rangle - \langle \hat{n}_{\text{loc}}(x_0) \rangle^2. \tag{2}$$

You should have seen similar expressions for e.g. position uncertainty ΔX in basic QM. We also define

$$p_{\text{loc}} = \int_{x_0}^{x_0+L} dx |\varphi_0(x)|^2, \tag{3}$$

which is the local probability to find an atom near x_0 in state 0.

(i) Assume the many-body quantum state is $\psi = |\, N, 0, 0, 0 \cdots \rangle$, i.e. all N atoms are in the ground state. Show that the mean local number in that state is $\langle \hat{n}_{\text{loc}}(x_0) \rangle = N p_{\text{loc}}$.

(ii) Then show that the local number uncertainty in (2) is $N(p_{\text{loc}} - p_{\text{loc}}^2)$.

(iii) Redo the two steps above for the state $\psi = [|\, N-k, 0, 0, 0 \cdots \rangle + |\, N+k, 0, 0, 0 \cdots \rangle]/\sqrt{2}$.

(iv) Think about the result for part (ii) using just statistical arguments, not quantum mechanics.

(2) Coherent states and Wigner functions:

(i) Show the Campbell-Baker-Hausdorff formula: Let us asume two operators $\hat{A}$ and $\hat{B}$ that may not commute $[\hat{A}, \hat{B}] \neq 0$. However assume we know that they both commute with their commutator, which means you can use $[\hat{A}, [\hat{A}, \hat{B}]] = 0$ and $[\hat{B}, [\hat{A}, \hat{B}]] = 0$). One example is when the commutator itself is just a number, e.g. $[\hat{A}, \hat{B}] \in \mathbb{C}$.

Show that then:

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}. \tag{4}$$

(ii) Using this, show that the characteristic function (see lecture notes Eq. 2.45) of a coherent state ($\hat{\rho} = |\,\beta\,\rangle\langle\,\beta\,|$) is

$$\chi_W(\lambda, \lambda^*) = e^{-\frac{|\lambda|^2}{2}+\lambda\beta^*-\lambda^*\beta}. \tag{5}$$

(iii) Then perform the Fourier transform of this, to show that the Wigner function $W(\alpha, \alpha^*)$ is a Gaussian, centered on the complex number β:

$$W(\alpha, \alpha^*) = 2 \exp\left[-2|\lambda - \beta|^2\right]. \tag{6}$$

Hint: To realize that you have to do a Fourier Transform, you may have to split β and λ into real and imaginary parts. Feel free to use Mathematica for any assignment question. If you do, include a printout of the .nb file in your solution

(3) Numerical evaluation of Wigner function: [4 points]

(i) Let us present the Fock space for a single mode for a restricted maximum number of particles $N_{\max}$ through a vector in $\mathbb{R}^{N_{\max}+1}$. This means for $N_{\max} = 2$, $|\,0\,\rangle \to [1,0,0]^T$, $|\,1\,\rangle \to [0,1,0]^T$, $|\,2\,\rangle \to [0,0,1]^T$. Using this, write down a matrix representation for the creation and destruction operators.

(ii) Using the same, write down the matrix representation for the density matrix in a Fock state $|\,n\,\rangle$ or a coherent state $|\,\alpha\,\rangle$.

(iii) Combine these two results, to adjust the template matlab script `Assignment2_wignerfct_v1.m`, such that it can plot the Wigner function of an arbitrary state. Use it first to confirm your result of (2) for a coherent state. Then plot a Fock state. Then plot some other interesting states of your choice.