PHY635, I-Semester 2019/20, Assignment 2, solution

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(1) Ideal Bose gas, density fluctuations: [In this question, supply the details for the example in section 2.3.1]. Consider N Bosonic atoms in a 1D harmonic trap. To measure local density, we count atoms in a small region of size L, which corresponds to the operator

$$\hat{n}_{\text{loc}}(x_0) = \int_{x_0}^{x_0+L} dx \hat{\Psi}^\dagger(x)\hat{\Psi}(x), \tag{1}$$

and then use $\hat{\rho} = n_{\text{loc}}(x_0)/L$ to get a density.

Let us define the <u>local number uncertainty</u>

$$\Delta n_{\text{loc}}(x_0)^2 = \langle \hat{n}_{\text{loc}}(x_0)^2 \rangle - \langle \hat{n}_{\text{loc}}(x_0) \rangle^2. \tag{2}$$

You should have seen similar expressions for e.g. position uncertainty ΔX in basic QM. We also define

$$p_{\text{loc}} = \int_{x_0}^{x_0+L} dx |\varphi_0(x)|^2, \tag{3}$$

which is the local probability to find an atom near x_0 in state 0.

(i) Assume the many-body quantum state is $\psi = |\, N, 0, 0, 0 \cdots \rangle$, i.e. all N atoms are in the ground state. Show that the mean local number in that state is $\langle \hat{n}_{\text{loc}}(x_0) \rangle = N p_{\text{loc}}$.

(ii) Then show that the local number uncertainty in (2) is $N(p_{\text{loc}} - p_{\text{loc}}^2)$.

(iii) Redo the two steps above for the state $\psi = [|\, N-k, 0, 0, 0 \cdots \rangle + |\, N+k, 0, 0, 0 \cdots \rangle]/\sqrt{2}$.

(iv) Think about the result for part (ii) using just statistical arguments, no quantum mechanics.

Solution:

(i) The mean local number in the given state is,

$$
\begin{aligned}
\langle \hat{n}_{loc}(x_0)\rangle &= \langle N,0,0\cdots | \int_{x_0}^{x_0+L} dx \hat{\Psi}^\dagger(x)\hat{\Psi}(x) | N,0,0\cdots\rangle \\
&= \int_{x_0}^{x_0+L} dx \langle N,0,0\cdots | \sum_n \sum_m \varphi_n^*(x)\varphi_m(x)\hat{a}_n^\dagger \hat{a}_m | N,0,0\cdots\rangle \\
&= \int_{x_0}^{x_0+L} dx \varphi_0^*(x)\varphi_0(x)\langle N,0,0\cdots |\hat{a}_0^\dagger \hat{a}_0| N,0,0\cdots\rangle \\
&= N\int_{x_0}^{x_0+L} dx |\varphi_0(x)|^2 \\
&= Np_{loc}.
\end{aligned} \tag{4}
$$

In the third equality we have used, that any other combination of creation and destruction operators will give zero if sandwiched in that particular Fock-state.

(ii) The local number uncertainty is,

$$
\begin{aligned}
\Delta n_{loc}(x_0)^2 &= \langle \hat{n}_{loc}(x_0)^2\rangle - \langle \hat{n}_{loc}(x_0)\rangle^2 \\
&= \langle N,0,0\cdots | \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \hat{\Psi}^\dagger(x)\hat{\Psi}(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x') | N,0,0\cdots\rangle \\
&- \left(\langle N,0,0\cdots | \int_{x_0}^{x_0+L} dx \hat{\Psi}^\dagger(x)\hat{\Psi}(x) | N,0,0\cdots\rangle\right)^2
\end{aligned} \tag{5}
$$

Writing the first term of Eq. (5) in normal order form,

$$
\begin{aligned}
\Delta n_{loc}(x_0)^2 &= \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \langle N,0,0\cdots |\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x)\hat{\Psi}(x')| N,0,0\cdots\rangle \\
&+ \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \langle N,0,0\cdots |\hat{\Psi}^\dagger(x)\delta(x-x')\hat{\Psi}(x')| N,0,0\cdots\rangle \\
&- \underbrace{\left(\langle N,0,0\cdots | \int_{x_0}^{x_0+L} dx \hat{\Psi}^\dagger(x)\hat{\Psi}(x) | N,0,0\cdots\rangle\right)^2}_{=N^2p_{loc}^2\text{ , from (i)}} \\
&= \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \sum_{m,n,l,k} \varphi_m^*(x)\varphi_n^*(x')\varphi_l(x)\varphi_k(x')\langle N,0,0\cdots |\hat{a}_m^\dagger \hat{a}_n^\dagger \hat{a}_l \hat{a}_k| N,0,0\cdots\rangle \\
&+ \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \sum_{m,k} \varphi_m^*(x)\delta(x-x')\varphi_k(x')\langle N,0,0\cdots |\hat{a}_m^\dagger \hat{a}_k| N,0,0\cdots\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- N^2 p_{loc}^2 \\
&= \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \varphi_0^*(x)\varphi_0^*(x')\varphi_0 x)\varphi_0(x') \langle\, N,0,0\cdots|\hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0|\, N,0,0\cdots\rangle \\
&+ \int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \varphi_0^*(x)\delta(x-x')\varphi_0(x') \langle\, N,0,0\cdots|\hat{a}_0^\dagger \hat{a}_0|\, N,0,0\cdots\rangle \\
&- N^2 p_{loc}^2 \\
&= N(N-1)p_{loc}^2 + N p_{loc} - N^2 p_{loc}^2 \\
&= N(p_{loc} - p_{loc}^2) \qquad (6)
\end{aligned}
$$

In the third equality, we have used that any other combination of creation and destruction operators will give zero if sandwiched in that particular Fock-state. Without going to normal order in the first step, we could not have *used this argument, since e.g.* $\langle\, N,0,0\cdots|\hat{a}_0^\dagger \hat{a}_n \hat{a}_n^\dagger \hat{a}_0|\, N,0,0\cdots\rangle \neq 0$ *for all n.*

(iii) For the other given state the mean local number is,

$$
\begin{aligned}
\langle \hat{n}_{loc}(x_0)\rangle &= \Big\{ \langle\, N-k,0,0\cdots|\int_{x_0}^{x_0+L} dx \hat{\Psi}^\dagger(x)\hat{\Psi}(x)|\, N-k,0,0\cdots\rangle \\
&+ \langle\, N+k,0,0\cdots|\int_{x_0}^{x_0+L} dx \hat{\Psi}^\dagger(x)\hat{\Psi}(x)|\, N+k,0,0\cdots\rangle \Big\}/2 \\
&= \Big\{ \int_{x_0}^{x_0+L} dx \langle\, N-k,0,0\cdots|\sum_n\sum_m \varphi_n^*(x)\varphi_m(x)\hat{a}_n^\dagger \hat{a}_m|\, N-k,0,0\cdots\rangle \\
&+ \int_{x_0}^{x_0+L} dx \langle\, N+k,0,0\cdots|\sum_n\sum_m \varphi_n^*(x)\varphi_m(x)\hat{a}_n^\dagger \hat{a}_m|\, N+k,0,0\cdots\rangle \Big\}/2 \\
&= \Big\{ \int_{x_0}^{x_0+L} dx \varphi_0^*(x)\varphi_0(x) \langle\, N-k,0,0\cdots|\hat{a}_0^\dagger \hat{a}_0|\, N-k,0,0\cdots\rangle \\
&+ \int_{x_0}^{x_0+L} dx \varphi_0^*(x)\varphi_0(x) \langle\, N+k,0,0\cdots|\hat{a}_0^\dagger \hat{a}_0|\, N+k,0,0\cdots\rangle \Big\}/2 \\
&= \Big\{ (N-k)\int_{x_0}^{x_0+L} dx |\varphi_0(x)|^2 + (N+k)\int_{x_0}^{x_0+L} dx |\varphi_0(x)|^2 \Big\}/2 \\
&= N p_{loc}. \qquad (7)
\end{aligned}
$$

The local number uncertainty in the given state is,

$$
\begin{aligned}
\Delta n_{loc}(x_0)^2 &= \langle \hat{n}_{loc}(x_0)^2\rangle - \langle \hat{n}_{loc}(x_0)\rangle^2 \\
&= \Big\{ \langle\, N-k,0,0\cdots|\int_{x_0}^{x_0+L} dx \int_{x_0}^{x_0+L} dx' \hat{\Psi}^\dagger(x)\hat{\Psi}(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x')|\, N-k,0,0\cdots\rangle
\end{aligned}
$$

$$+\langle N+k,0,0\cdots|\int_{x_0}^{x_0+L}dx\int_{x_0}^{x_0+L}dx'\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x')|N+k,0,0\cdots\rangle\Big\}/2$$

$$-\Big(\Big\{\langle N-k,0,0\cdots|\int_{x_0}^{x_0+L}dx\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|N-k,0,0\cdots\rangle$$

$$+\langle N+k,0,0\cdots|\int_{x_0}^{x_0+L}dx\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|N+k,0,0\cdots\rangle\Big\}/2\Big)^2 \quad (8)$$

writing the first term of Eq. (8) in normal order form,

$$\Delta n_{loc}(x_0)^2=\Big\{\int_{x_0}^{x_0+L}dx\int_{x_0}^{x_0+L}dx'\langle N-k,0,0\cdots|\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x)\hat{\Psi}(x')|N-k,0,0\cdots\rangle$$

$$+\int_{x_0}^{x_0+L}dx\int_{x_0}^{x_0+L}dx'\langle N+k,0,0\cdots|\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x)\hat{\Psi}(x')|N+k,0,0\cdots\rangle\Big\}/2$$

$$+\Big\{\int_{x_0}^{x_0+L}dx\int_{x_0}^{x_0+L}dx'\langle N-k,0,0\cdots|\hat{\Psi}^\dagger(x)\delta(x-x')\hat{\Psi}(x')|N-k,0,0\cdots\rangle$$

$$+\int_{x_0}^{x_0+L}dx\int_{x_0}^{x_0+L}dx'\langle N+k,0,0\cdots|\hat{\Psi}^\dagger(x)\delta(x-x')\hat{\Psi}(x')|N+k,0,0\cdots\rangle\Big\}/2$$

$$-\Big(\Big\{\langle N-k,0,0\cdots|\int_{x_0}^{x_0+L}dx\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|N-k,0,0\cdots\rangle$$

$$+\langle N+k,0,0\cdots|\int_{x_0}^{x_0+L}dx\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|N+k,0,0\cdots\rangle\Big\}/2\Big)^2 \quad (9)$$

Applying the formalism as in Eq. (6) we obtain:

$$\Delta n_{loc}(x_0)^2=\frac{\{(N-k)(N-k-1)+(N+k)(N+k-1)\}p_{loc}^2}{2}$$

$$+\frac{\{(N+k)+(N-k)\}p_{loc}}{2}-N^2p_{loc}^2$$

$$=N(p_{loc}-p_{loc}^2)+k^2p_{loc}^2. \quad (10)$$

(iv) *Consider N classical atoms distributed randomly in space with probability distribution function $|\varphi_0(x)|^2$. For a single atom the probability to be in the interval $[x_0,x_0+L]$ is thus p_{loc}. The probability to find exactly m atoms in that interval is now:*

$$P_m=p_{loc}^m(1-p_{loc})^{(N-m)}.\binom{N}{m} \quad (11)$$

The first two factors are the probability that the first m atoms are in the interval and the last $N-m$ outside. However since we don't care about which *atoms are in* *the interval, we multiply this by the Binomial coefficient giving the number of ways to draw m out of N. We reckognize Eq. (11) as the Binomial distribution, which has a variance of $Np_{loc}(1 - p_{loc})$, in agreement with (ii).*

(2) Coherent states and Wigner functions:

(i) *Show the Campbell-Baker-Hausdorff formula: For two operators $\hat{A}$ and $\hat{B}$ that may not commute $\left[\hat{A}, \hat{B}\right] \neq 0$, but that commute with their commutator ($\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] = 0$ and $\left[\hat{B}, \left[\hat{A}, \hat{B}\right]\right] = 0$), we have*

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}\left[\hat{A},\hat{B}\right]}. \tag{12}$$

(ii) *Using this, show that the characteristic function (see lecture notes Eq. 2.45) of a coherent state ($\hat{\rho} = |\,\beta\,\rangle\langle\,\beta\,|$) is*

$$\chi_W(\lambda, \lambda^*) = e^{-\frac{|\lambda|^2}{2}+\lambda\beta^*-\lambda^*\beta}. \tag{13}$$

(iii) *Then perform the Fourier transform of this, to show that the Wigner function $W(\alpha, \alpha^*)$ is a Gaussian, centered on the complex number β:*

$$W(\alpha, \alpha^*) = 2\exp\left[-2|\lambda - \beta|^2\right]. \tag{14}$$

Hint: To realize that you have to do a Fourier Transform, you may have to split β and λ into real and imaginary parts. Feel free to use Mathematica for any assignment question. If you do, include a printout of the .nb file in your solution

Solution:

(i) *You should be able to find many proofs in the literature. The following is one variant. To obtain the formula we define*

$$e^{xA}e^{xB} = e^{H(x)} = e^{xH_1+x^2H_2+x^3H_3+\cdots} \tag{15}$$

and consider

$$e^{xA}e^{xB}\frac{d}{dx}e^{-xA}e^{-xB} = e^{-H(x)}\frac{d}{dx}e^{H(x)} \tag{16}$$

The l.h.s. of Eq. (16) can be written as,

$$\begin{aligned} e^{xA}e^{xB}\frac{d}{dx}e^{-xA}e^{-xB} &= e^{-xB}Be^{xB} + e^{-xB}e^{-xA}Ae^{-xA}e^{xB} \\ &= B + e^{-xB}Ae^{-xB} \\ &= B + A + x[A, B] + \frac{x^2}{2}[B, [B, A]] + \cdots \end{aligned} \tag{17}$$

while the r.h.s. of Eq. (16) can be written as,

$$e^{-H(x)}\frac{d}{dx}e^{H(x)} = H' + \frac{1}{2}[H', H] + \frac{1}{3!}[[H', H], H] + \cdots$$
$$= H_1 + 2xH_2 + x^2\left(3H_3 - \frac{1}{2}[H_1, H_2]\right) + O(x^3) \qquad (18)$$

where we have used

$$[H', H] = [H_1 + 2xH_2 + 3x^2H_3 + \cdots, xH_1 + x^2H_2 + x^3H_3 + \cdots]$$
$$= x^2[H_1, H_2] + 2x^2[H_2, H_1] + O(x^3)$$
$$= -x^2[H_1, H_2] + O(x^3) \qquad (19)$$

Equating Eq. (17) and Eq. (18) to $O(x^2)$ we obtain

$$H_1 = A + B$$
$$H_2 = \frac{1}{2}[A, B]$$
$$H_3 = \frac{1}{12}\left([A, [A, B]] + [B, [B, A]]\right) \qquad (20)$$

Using Eq. (20) in Eq. (15) (set $x = 1$) we obtain

$$e^A e^B = e^{A+B+\frac{1}{2}[A+B]} \qquad (21)$$

(ii) The characteristic function of a coherent state is defined as,

$$\chi_W(\lambda, \lambda^*) = tr\{\hat{\rho}e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}\}. \qquad (22)$$

Using the Campbell-Baker-Hausdorff formula we can write Eq. (22) as,

$$\chi_W(\lambda, \lambda^*) = tr\{\hat{\rho}e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}e^{-|\lambda|^2/2}\}$$
$$= e^{-|\lambda|^2/2}\langle\, \beta\, |e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}|\, \beta\, \rangle$$
$$= e^{-|\lambda|^2/2}\langle\, \beta\, |\sum_n \frac{(\lambda\hat{a}^\dagger)^n}{n!}\sum_m \frac{(-\lambda^*\hat{a})^m}{m!}|\, \beta\, \rangle$$
$$= e^{-|\lambda|^2/2}\sum_n \frac{(\lambda\beta^*)^n}{n!}\sum_m \frac{(-\lambda^*\beta)^m}{m!}$$
$$= e^{-|\lambda|^2/2}e^{\lambda\beta^*}e^{-\lambda^*\beta}$$
$$= e^{-|\lambda|^2/2+\lambda\beta^*-\lambda^*\beta}. \qquad (23)$$

(iii) *The Wigner distribution is defined as,*

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda e^{-\lambda\alpha^* + \lambda^*\alpha} \chi_W(\lambda, \lambda^*). \quad (24)$$

Inserting the form of χ_W from Eq. (23) we obtain

$$\begin{aligned} W(\alpha, \alpha^*) &= \frac{1}{\pi^2} \int d^2\lambda e^{-\lambda\alpha^* + \lambda^*\alpha} e^{-|\lambda|^2/2 + \lambda\beta^* - \lambda^*\beta} \\ &= \frac{1}{\pi^2} \int d^2\lambda e^{\lambda(-\alpha^* + \beta^*)} e^{\lambda^*(\alpha - \beta)} e^{-|\lambda|^2/2} \\ &= \frac{1}{\pi^2} \int d^2\lambda \sum_{m,n} \frac{\lambda^m(-\alpha^* + \beta^*)^m}{m!} \frac{(\lambda^*)^n(\alpha - \beta)^n}{n!} e^{-|\lambda|^2/2}. \end{aligned} \quad (25)$$

Now, Writing λ in polar from as,

$$\lambda = re^{i\phi} \implies d^2\lambda = rdrd\phi \quad (26)$$

we get

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \sum_{m,n} \int_0^{2\pi} d\phi e^{i(m-n)\phi} \int_0^{\infty} drr^{m+n+1} e^{-r^2/2} \frac{(-\alpha^* + \beta^*)^m}{m!} \frac{(\alpha - \beta)^n}{n!}. \quad (27)$$

After integrating we get

$$\begin{aligned} W(\alpha, \alpha^*) &= \frac{2}{\pi} \sum_m 2^m \frac{(-\alpha^* + \beta^*)^m (\alpha - \beta)^m}{m!} \\ &= \frac{2}{\pi} \sum_m (-1)^m \frac{2^m |\alpha - \beta|^{2m}}{m!} \\ &= \frac{2}{\pi} e^{-2|\alpha - \beta|^2} \end{aligned} \quad (28)$$

(3) Numerical evaluation of Wigner function: [4 points]

(i) *Let us present the Fock space for a single mode for a restricted maximum number of particles N_{max} through a vector in $\mathbb{R}^{N_{max}+1}$. This means for $N_{max} = 2$, $|0\rangle \to [1,0,0]^T$, $|1\rangle \to [0,1,0]^T$, $|2\rangle \to [0,0,1]^T$. Using this, write down a matrix representation for the creation and destruction operators.*

(ii) *Using the same, write down the matrix representation for the density matrix in a Fock state $|n\rangle$ or a coherent state $|\alpha\rangle$.*

(iii) *Combine these two results, to adjust the template matlab script `Assignment2_wignerfct_v1.m`, such that it can plot the Wigner function of an arbitrary state. Use it first to confirm your result of (2) for a coherent state. Then plot a Fock state. Then plot some other interesting states of your choice.*

Solution:

(i) The matrix representation of the destruction operator is,

$$a = \begin{pmatrix} \langle\, 0\,|\hat{a}|\, 0\,\rangle & \langle\, 0\,|\hat{a}|\, 1\,\rangle & \langle\, 0\,|\hat{a}|\, 2\,\rangle & \cdots & \langle\, 0\,|\hat{a}|\, N_{max}\,\rangle \\ \langle\, 1\,|\hat{a}|\, 0\,\rangle & \cdots & \cdots & \cdots & \cdots \\ \langle\, 2\,|\hat{a}|\, 0\,\rangle & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\, N_{max}\,|\hat{a}|\, 0\,\rangle & \langle\, N_{max}\,|\hat{a}|\, 1\,\rangle & \langle\, 0\,|\hat{a}|\, N_{max}\,\rangle & \cdots & \langle\, N_{max}\,|\hat{a}|\, N_{max}\,\rangle \end{pmatrix}$$

$$a = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{N_{max}} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The matrix representation of the creation operator is,

$$a^{\dagger} = \begin{pmatrix} \langle\, 0\,|\hat{a}^{\dagger}|\, 0\,\rangle & \langle\, 0\,|\hat{a}^{\dagger}|\, 1\,\rangle & \langle\, 0\,|\hat{a}^{\dagger}|\, 2\,\rangle & \cdots & \langle\, 0\,|\hat{a}^{\dagger}|\, N_{max}\,\rangle \\ \langle\, 1\,|\hat{a}^{\dagger}|\, 0\,\rangle & \cdots & \cdots & \cdots & \cdots \\ \langle\, 2\,|\hat{a}^{\dagger}|\, 0\,\rangle & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\, N_{max}\,|\hat{a}^{\dagger}|\, 0\,\rangle & \langle\, N_{max}\,|\hat{a}^{\dagger}|\, 1\,\rangle & \langle\, 0\,|\hat{a}^{\dagger}|\, N_{max}\,\rangle & \cdots & \langle\, N_{max}\,|\hat{a}^{\dagger}|\, N_{max}\,\rangle \end{pmatrix}$$

$$a^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \sqrt{N_{max}} & 0 \end{pmatrix}$$

(ii) The given state can be written as,

$$|\, n\,\rangle = \sum_{l=0}^{N_{max}} c_l |\, l\,\rangle. \tag{29}$$

The density matrix can be given as,

$$\begin{aligned}\hat{\rho} &= |\, n \,\rangle\langle\, n \,| \\ &= \sum_{m=0}^{N_{max}} \sum_{l=0}^{N_{max}} c_l^* c_m |\, l \,\rangle\langle\, m \,| \end{aligned} \tag{30}$$

Now the matrix representation is given as:

$$\rho = \begin{pmatrix} \langle\, 0 \,|\hat{\rho}|\, 0 \,\rangle & \langle\, 0 \,|\hat{\rho}|\, 1 \,\rangle & \langle\, 0 \,|\hat{\rho}|\, 2 \,\rangle & \cdots & \langle\, 0 \,|\hat{\rho}|\, N_{max} \,\rangle \\ \langle\, 1 \,|\hat{\rho}|\, 0 \,\rangle & \cdots & \cdots & \cdots & \cdots \\ \langle\, 2 \,|\hat{\rho}|\, 0 \,\rangle & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\, N_{max} \,|\hat{\rho}|\, 0 \,\rangle & \langle\, N_{max} \,|\hat{\rho}|\, 1 \,\rangle & \langle\, 0 \,|\hat{\rho}|\, N_{max} \,\rangle & \cdots & \langle\, N_{max} \,|\hat{\rho}|\, N_{max} \,\rangle \end{pmatrix}$$

$$\rho = \begin{pmatrix} |c_0|^2 & c_0^* c_1 & c_0^* c_2 & \cdots & \cdots & c_0^* c_{N_{max}} \\ c_1^* c_0 & |c_1|^2 & c_1^* c_2 & \cdots & \cdots & c_1^* c_{N_{max}} \\ c_2^* c_0 & c_2^* c_1 & |c_2|^2 & \cdots & \cdots & c_2^* c_{N_{max}} \\ c_3^* c_0 & c_3^* c_1 & c_3^* c_2 & |c_3|^2 & \cdots & c_3^* c_{N_{max}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{N_{max}}^* c_0 & c_{N_{max}} c_1 & c_{N_{max}}^* c_2 & \cdots & \cdots & |c_{N_{max}}|^2 \end{pmatrix}$$

(iii)

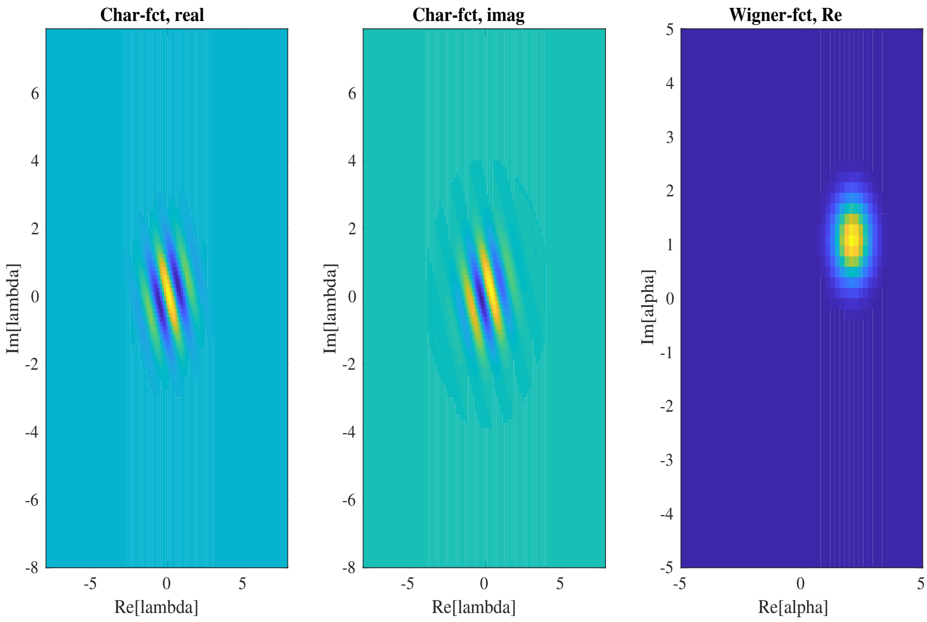

Fig. 1. Characteristic function and Wigner distribution for the result in question number 2.