

Week 0 / 1

PHY 635 Many-body Quantum Mechanics of Degenerate Gases

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These notes are provided for the students of the class above only. There is no warranty for correctness, please contact me if you spot a mistake.

0 Administrative affairs

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(ii) Literature: No Single Textbook

- Pethick and Smith, “*Bose-Einstein Condensation in Dilute Gases*” [**closest matching book**]
- R. Shankar, “*Principles of Q.M*”
- J.J. Sakurai, “*Modern Quantum Mechanics*”
- Negele and Orland, “*Quantum Many Particle Systems*”
- Petter and Walecka, “*Quantum Theory of Many Particle Systems*”
- L. Schiff, “*Quantum Mechanics*”
- Henrik Bruus, Flensberg, “*Introduction to Many Body Quantum Mechanics in Condensed Matter*”

There is no primary text-book. I collect material from wherever I find it best covered case-by-case.

(iii) Assessment:

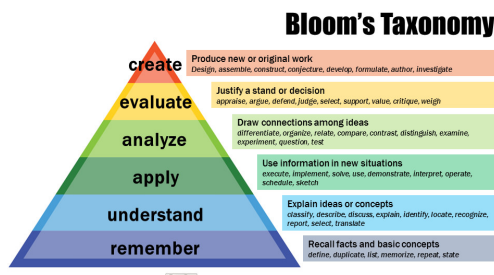
- **Three scheduled Quizzes in the AIR studio with examineer: 15%** To supplement the exams, there will be three quizzes lasting one lecture hour, conducted using [the examineer webpage](#) . For quiz timings see webpage. These are ”open notes” quizzes, so bring a copy of your notes. Quizzes will be simpler than exams, intended to encourage you to continuously keep on top of the material. **Make sure to bring an internet capable phone/tablet/laptop on the days of the quiz**, if you use online notes a **second** device for the notes. Please contact me if there is a problem with this. You also need some pen and paper for scrap notes on quiz days. There will also be a few, even easier ones that you can do from home over a few days.

- **Assignments: 15%** There will be about five assignments handed out with a two week deadline each. I expect you to form teams of 3-4 students and stick in these teams for the semester. Hand in only one solution per team. The TA is instructed to give full marks for *any serious attempt* at a given question of the assignment, even if the result is wrong. This is to discourage copying and encourage doing it yourself. Additionally however the TA is asked to deduct marks for messy presentation and blatant copying. These assignments are intended as your primary means of learning the essential course material, please do all of them diligently.
- **Numerics component of assignments:** Moderns science almost always necessitates the heavy use of computers. Most assignments will contain a numerics component, to be done using [XMDS](#) . I will provide a tool and template package that you have to only minorly edit. See notes on numerics assignments online.
- **Mid-Sem exam: 30%**
- **Final exam: 40%** The exams will try to test understanding of the essential *physics* concepts taught, not maths. For guidance regarding what are the most important concepts look at the quizzes and assignments. All exams will be designed to give a significant advantage to those students that solved all assignments by *themselves*.

Course outline

- 1) Motivation and Review: ~ 1 week
 - Fields that require Many-Body QM, why degenerate gases?
 - Review essential pieces from single-particle QM
- 2) Quantum-many-body Formalism: $\sim 4-5$ weeks
 - Second Quantisation, Bosons vs Fermions, quantum field operators, coherent states
- 3) Bose-Einstein Condensates (BECs): $\sim 4-5$ weeks
 - Symmetry breaking/ mean field, Critical Temperature, Gross- Pitaevskii equation, Bogoliubov quasi-particles, Quantum field theory of BEC.
- 4) Degenerate Femi Gases $\sim 3-4$ weeks
 - Fermi Surface, Degeneracy Pressure, Neutron stars, electron gas, pairing, superconductivity
- 5) *Online(?)*: Quantum Simulators: $\sim 2-3$ weeks
 - Analog and digital quantum Simulation, Bose-Hubbard model, BEC-BCS cross-over

Learning objectives



left: Research on learning has classified possible targets in the pyramid on the left.

In terms of skills in the pyramid on the top, this course aims to foster levels 2 and 3 (understand,

apply). This means also the exam will be designed to test these. In particular you should not be able to score a lot with only level 1 (remembering), since nowadays remembering is not that crucial thanks to the internet. After the course I would be grateful for feedback whether this course/exam design goal has been reached.

1 Motivation and Review

1.1 Quantum many-body theory and Quantum field theory (QFT)

Single quantum particle can be described by a state $\varphi(\mathbf{x})$.

Many Particle Wavefunctions: are more complicated.

- We have to firstly take care of Bosonic/Fermionic Symmetries:

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2}} [\varphi_1(\mathbf{x}_1) \varphi_2(\mathbf{x}_2) \pm \varphi_1(\mathbf{x}_2) \varphi_2(\mathbf{x}_1)] \quad (1.1)$$

- Then we often have to worry about particle creation or destruction $\psi(x_1) \leftrightarrow \psi(x_1, x_2) \leftrightarrow \psi(x_1, x_2, x_3) \leftrightarrow \dots$
- Very helpful for this is Second Quantisation with creation and destruction operators: \hat{a}, \hat{a}^\dagger (same algebra as Simple Harmonic oscillator ladder operators)
- The natural next step from there is Quantum field theory

$$\hat{\Psi}(\mathbf{x}) = \sum_n \underbrace{\varphi_n(\mathbf{x})}_{\text{wave aspect}} \underbrace{\hat{a}_n}_{\text{particle aspect}} \quad (1.2)$$

Where $\hat{\Psi}$ can be viewed as a type of destruction operator

- Compared to a classical field \rightarrow now the field itself is quantised
- For $v \approx c$ we need to do relativistic Quantum Mechanics and then relativistic Quantum Field theory as used in elementary particle physics. We do not cover this, but will mention the Spin-Statistics theorem, which originates from there.
- Condensed matter physics provides lots of advanced QFT techniques, such as Greens-functions, Path-Integrals, Keldysh formalism or thermal field theory, we also do not require these.

1.2 Disciplines with many-body QM

Many sub-fields of physics require us to tackle many-body quantum mechanics.

- Atomic and Molecular Physics (many e^-)
- Chemistry (even more e^-)
- Nuclear Physics (many n,p)
- Particle Physics (several elementary particles, but usually more concerned with creation/ destruction/ conversion/ symmetries than with the number of particles)
- Condensed Matter Physics (10^{23} e^- or Quasiparticles like Spinon, Magnon, Plasmon etc.)

- Astrophysics (even more e^-/n)

Recently:

- Quantum Optics (many γ , usually non-interacting)
- Quantum Information (many 0,1)

1.3 Review

1.3.1 Single-particle States

In this course we will always assume the problem of a single particle to be solved, e.g with the

Time-independent Schrödinger equation (TISE)

$$\hat{H}_0|\varphi_n\rangle = E_n|\varphi_n\rangle. \quad (1.3)$$

- Where \hat{H}_0 is the single body Hamiltonian (depends on co-ordinates of just one particle)
- $|\varphi_n\rangle$ is the (typically) infinite single-particle basis. The φ_n are also called mode.
- E_n are single particle energies

Examples: We will require these to construct many-body theories later, so this is to set notation

(i) Free particles in Volume \mathcal{V}

$$\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2 \quad E_n = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m} \quad (1.4)$$

$$|\psi_n\rangle \rightsquigarrow |\phi_k\rangle = \frac{1}{\sqrt{\mathcal{V}}}e^{i\mathbf{k}\mathbf{x}} \quad (1.5)$$

where \mathbf{k} is the wavenumber. Technically we have to distinguish continuous and discrete spectra, where the free particle is continuous, see books. We will always use a discrete notation for simplicity.

(ii) Spin - $\frac{1}{2}$ states

$$\hat{H}_0 = \frac{\Delta E}{2}\hat{\sigma}_3 \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad E_{\uparrow\downarrow} = \pm \frac{\Delta E}{2} \quad (1.6)$$

The basis are spin-up and spin-down states $|\varphi_{\uparrow,\downarrow}\rangle$, specifically $|\varphi_{\uparrow}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cong |\uparrow\rangle$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cong |\downarrow\rangle$

Example cont.: (iii) Simple Harmonic oscillator states (3D), with Hamiltonian

$$H_0 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \quad (1.7)$$

3D Harmonic oscillator Quantum states

$$|\varphi_{\mathbf{n}}\rangle = |\varphi_{n_x n_y n_z}\rangle = \varphi_{n_x}(x)\varphi_{n_y}(y)\varphi_{n_z}(z) \quad (= |\varphi_{\mathbf{n}}(\mathbf{x})\rangle) \quad (1.8)$$

$$\varphi_{n_i}(x_i) = \frac{1}{\sqrt{2^{n_i} (n_i)! \sqrt{\pi} \sigma_i}} e^{-\frac{x_i^2}{2\sigma_i^2}} H_{n_i}\left(\frac{x_i}{\sigma_i}\right) \quad (1.9)$$

where $x_i \in \{x, y, z\}$, and $H_n(x)$ are Hermite polynomials. The oscillator widths in the three directions are $\sigma_i = \sqrt{\hbar/(m\omega_i)}$. Energies:

$$E_n = \hbar\omega_x(n_x + \frac{1}{2}) + \hbar\omega_y(n_y + \frac{1}{2}) + \hbar\omega_z(n_z + \frac{1}{2}). \quad (1.10)$$

The 3D eigenstates thus factor into the 1D eigenstates which you know, and which look like



left: Sketch of 1D harmonic oscillator potential and the first few eigenstates

(iv) Other examples: from PHY 303/304: particle in a 1D/2D/3D box potential, Single electron in a Hydrogen atom...

Going back to the 1D oscillator, we recall the

Ladder-Operators (1D)

Lowering operator $\hat{b} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i\sqrt{\frac{1}{2m\omega\hbar}} \hat{p} \quad (1.11)$

Raising operator $\hat{b}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i\sqrt{\frac{1}{2m\omega\hbar}} \hat{p} \quad (1.12)$

We can show that $[\hat{b}, \hat{b}^\dagger]=1$ (from $[\mathbf{x}, \mathbf{p}] = i\hbar$)

New Hamiltonian using ladder operators (from 1D version of (1.7)).

$$\hat{H}_0 = \hbar\omega \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \quad (1.13)$$

We can now show (e.g. in Shankar \cong pg. 204) the

Function of Raising and Lowering Operators

$$\hat{b}|\varphi_n\rangle = \sqrt{n}|\varphi_{n-1}\rangle \quad \hat{b}|\varphi_0\rangle = 0, \quad (1.14)$$

$$\hat{b}^\dagger|\varphi_n\rangle = \sqrt{n+1}|\varphi_{n+1}\rangle, \quad (1.15)$$

and define the phonon number Operator: $\hat{N} = \hat{b}^\dagger \hat{b}$ and $\hat{N}|\varphi_n\rangle = n|\varphi_n\rangle$.

- These properties follow solely from the commutation relation $[\hat{b}, \hat{b}^\dagger]=1$, and hence we would not need to know the position space representation (1.9).

1.3.2 Single Particle Density Matrices

Quantum mechanics allows superpositions of two “opposite” states, e.g. $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.

Classically we can already have random probability distributions of opposite states e.g. $P(\uparrow) = 50\%$, $P(\downarrow) = 50\%$ (let $\uparrow\downarrow$ be “head/tail” for a flipped coin here). The quantum superposition is a stronger statement, since it involves complex phases. However we require some mathematics to describe both sources of randomness simultaneously, since e.g. experiments might involve both. The mathematics is provided by the concept of the

Density Matrix/ operator

$$\hat{\rho} = \sum_{nm} \rho_{nm} |\varphi_n\rangle \langle \varphi_m|, \quad (1.16)$$

where $|\varphi_n\rangle$ is a chosen single particle basis as in section 1.3.1.

Density Matrix Properties

- For a pure quantum state: $\hat{\rho} = |\psi\rangle \langle \psi|$. So the statement “my system is in state $|\psi\rangle$ ” is equivalent to saying “my system has a density matrix $|\psi\rangle \langle \psi|$ ”.
- For a fully classical (mixed) state: $\rho_{nm} = 0$ for $n \neq m$
- Then $\hat{\rho} = \sum_n p_n |\varphi_n\rangle \langle \varphi_n|$ where p_n is the probability to be in state n .

- the expectation value of observable \hat{O} is now $\text{Tr}[\hat{O}\hat{\rho}]$, where $\text{Tr}[\dots]$ denotes the trace, so the sum of diagonal elements.
- $\hat{\rho}$ is Hermitian, for pure states $\hat{\rho}^2 = \hat{\rho}$, always $\text{Tr}[\hat{\rho}] = 1$
- We can define the purity P as $P = \text{Tr}[\hat{\rho}^2]$, then

$$P = 1 \quad \text{for pure states} \quad (1.17)$$

$$P < 1 \quad \text{for mixed states} \quad (1.18)$$

See your favourite QM book, the online notes for PHY435 “Decoherence and open quantum systems” and related literature on my webpage for a more detailed discussion of density matrices (in this order).

Time-Evolution of states involving density matrices is via the von-Neumann equation

$$i\hbar \frac{\partial}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}] \quad (1.19)$$

Examples:

Single spin- $\frac{1}{2}$ (see (ii) earlier)

(i) Let its quantum state be $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. The corresponding density matrix is $\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)$.

We can write this in Matrix form

$$\hat{\rho} = \begin{matrix} & |\uparrow\rangle & |\downarrow\rangle \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} & |\uparrow\rangle \\ & |\downarrow\rangle \end{matrix} \quad (1.20)$$

(ii) If we had a classical mixture instead (50 % \uparrow , 50% \downarrow), we would write $\hat{\rho} = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$, which becomes

$$\hat{\rho} = \begin{matrix} & |\uparrow\rangle & |\downarrow\rangle \\ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} & |\uparrow\rangle \\ & |\downarrow\rangle \end{matrix} \quad (1.21)$$

Diagonal Elements are called populations or probabilities. $\rho_{nn} \cong$ probability to find system in state n . **Off-diagonal Elements** are called coherences, where $\rho_{12} \cong$ indicates the amount of coherence between state 1 and 2, this is = 0 classically.

- Note that in both examples above, the probability for \uparrow and \downarrow is 50% each.
- **Distinction:** However we can discriminate the coherent superpositions from the classical

mixtures through the appearance of interference terms in certain observables, for example:

$$\langle \hat{\sigma}_x \rangle = \begin{cases} \langle \psi | \hat{\sigma}_x | \psi \rangle = \frac{\hbar}{2}, & \text{for case (i), pure} \\ \text{Tr} [\hat{\rho} \hat{\sigma}_x] = \text{Tr} \begin{bmatrix} 0 & \frac{\hbar}{2\sqrt{2}} \\ \frac{\hbar}{2\sqrt{2}} & 0 \end{bmatrix} = 0, & \text{for case (ii), mixed} \end{cases} \quad (1.22)$$

1.3.3 Many Particle States

The generalisation of section 1.3.1 to many particles is to add one co-ordinate per particle:

Many-body TISE (general case)

$$\hat{H}(\mathbf{x}_1, \dots, \mathbf{x}_n, \hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_n) \psi_k(\mathbf{x}_1, \dots, \mathbf{x}_n) = E_k \psi_k(\mathbf{x}_1, \dots, \mathbf{x}_n) \quad (1.23)$$

- Typically very high dimensional PDE (e.g. already 9D for 3 particles in 3-dimensions.)
- Often too high dimensional to deal with it directly → We learn techniques to sometimes deal with this problem in this lecture.
- k is an index numbering the eigenstate, as in single particle QM. But now the state is a many-body state. Often it makes sense to de-compose it into a collection of e.g. single particle indices, such as $k \equiv \{n_1, n_2, \dots, n_N\}$, see next dotpoint.
- We can always write many body states in terms of single-body ones, but the decomposition might be complicated:

$$\psi_k(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{n_1, \dots, n_N} c_{k; n_1, n_2, \dots, n_N} \varphi_{n_1}(\mathbf{x}_1) \varphi_{n_2}(\mathbf{x}_2) \dots \varphi_{n_N}(\mathbf{x}_N). \quad (1.24)$$

Orthogonality: Many body states as products of single particle basis are orthogonal, whenever any of the constituents differ, they inherit their orthogonality properties from the single particle states:

$$\begin{aligned} \langle \varphi_{n_1} | \langle \varphi_{n_2} | \langle \varphi_{n'_1} | \langle \varphi_{n'_2} \rangle &= \\ \int d^3 \mathbf{x}_1 d^3 \mathbf{x}_2 \varphi_{n_1}^*(\mathbf{x}_1) \varphi_{n_2}^*(\mathbf{x}_2) \varphi_{n'_1}(\mathbf{x}_1) \varphi_{n'_2}(\mathbf{x}_2) & \\ = \left(\int d^3 \mathbf{x}_1 \varphi_{n_1}^*(\mathbf{x}_1) \varphi_{n'_1}(\mathbf{x}_1) \right) \left(\int d^3 \mathbf{x}_2 \varphi_{n_2}^*(\mathbf{x}_2) \varphi_{n'_2}(\mathbf{x}_2) \right) &= \delta_{n_1 n'_1} \delta_{n_2 n'_2}. \end{aligned} \quad (1.25)$$

1.3.4 Entanglement

The generic many body states (1.24) can be classified as follows

Separable states: A many body state is called separable, if it can be written as a product of states for each particle:

$$\psi_{sep}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \prod_{i=1}^N \phi_{n_i}(\mathbf{x}_i) \quad (1.26)$$

All states that are not separable are called entangled.

- Less stringently, we can also talk about many-body states being separable or entangled with respect to two (or few) sub-systems A and B of the many-body Hilbertspace.

Examples: Separable:

$$|\psi\rangle = |\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \quad (1.27)$$

$$|\psi\rangle = \frac{1}{2} (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = \frac{1}{2} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \quad (1.28)$$

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\mathcal{V}}} e^{ik_1 x_1} \mathcal{N} \exp\left[-\frac{x_2^2}{2\sigma^2}\right] \quad (1.29)$$

Entangled:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \quad (1.30)$$

$$\psi(x_1, x_2) = \mathcal{N} \exp\left[-\frac{(x_1 - x_2)^2}{2\sigma_{\perp}^2} - \frac{(x_1 + x_2)^2}{2\sigma_{\parallel}^2}\right] \quad (1.31)$$

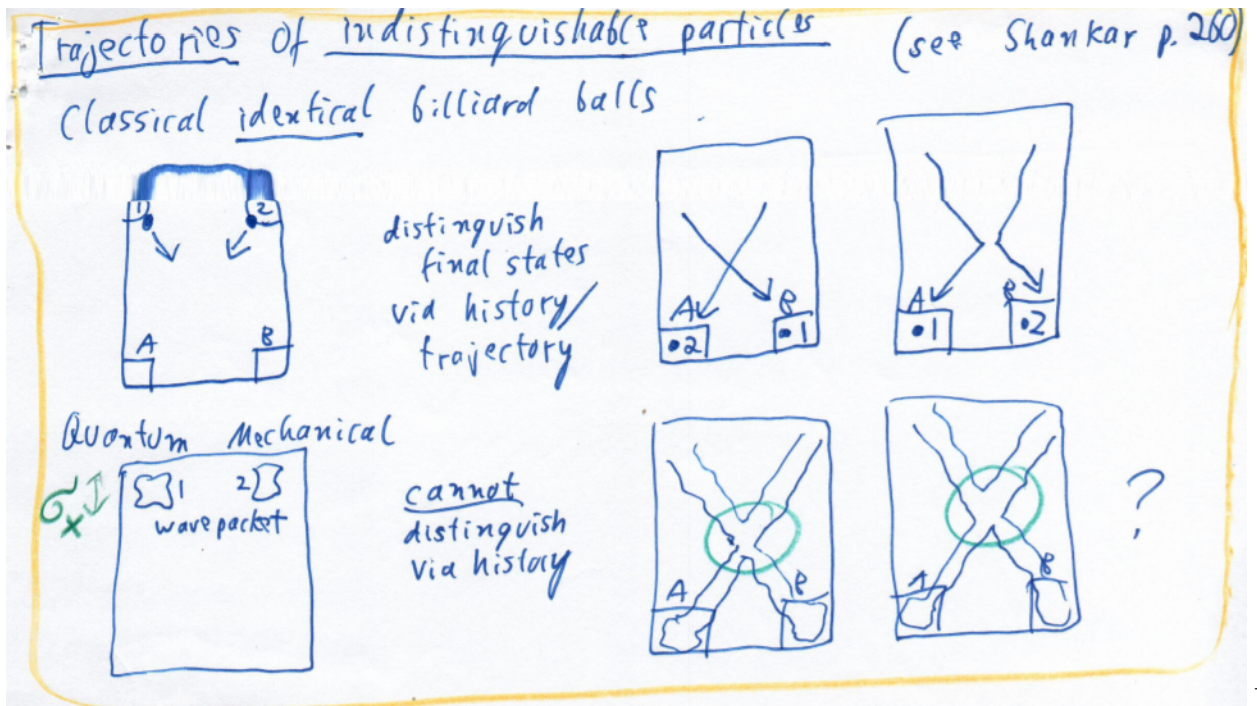
- In an entangled state (for two systems A,B) if we measure state A we typically know also about B.
- Entanglement also implies classical correlations, but it is much more than that (keywords: EPR paradox, Bell-theorem).
- The definition gets a bit more complicated for mixed states ($\hat{\rho}$).

1.3.5 Indistinguishable Particles

So far we have implicitly treated our many particles as distinguishable. For example the state (1.29) implies, “particle 1 is in a momentum eigenstate with momentum $\hbar k_1$, and particle two has a Gaussian wavefunction with width σ ”. However:

- A statement like, “Particle 1 is in state n_1 and Particle 2 is in state n_2 ” makes sense if the particles are distinguishable (say 1 = e^- (electron), 2 = p (proton), 3 = γ (photon)).

- For indistinguishable particles, the uncertainty relation forces us to abandon the labels “Particle 1”, “Particle 2” etc. (See box for “trajectories” below).
- This implies that mathematically, the state $\psi(\mathbf{x}_a, \mathbf{x}_b)$ must be equivalent to $\psi(\mathbf{x}_b, \mathbf{x}_a)$.



Since the overall phase or sign of a wave-function does not matter, “being equivalent” still allows for

Identical particle exchange symmetry

$$\psi(x_a, x_b) = \pm \psi(x_b, x_a) \quad (1.32)$$

\oplus = Bosons = Symmetric under exchange of any two indistinguishable particles

\ominus = Fermions = Anti-symmetric under exchange of any two indistinguishable particles.

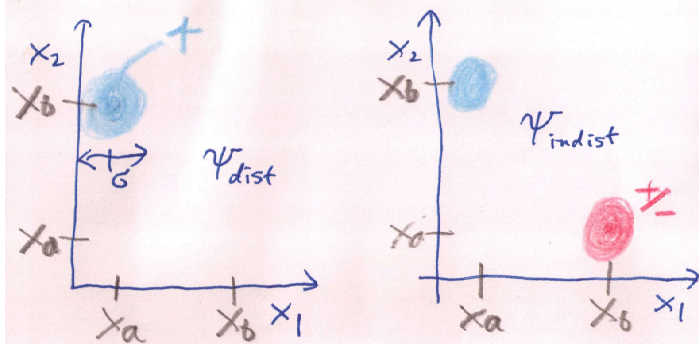
- The most important consequence of this is the Pauli exclusion principle: **Two indistinguishable Fermions cannot be in the same single particle state.** *Proof: Try to write a contradicting state down.*
- The need for special treatment of indistinguishable particles comes from the **green encircled region in the above figure** i.e when the matter waves overlap. If this never happens, particles can be tracked and (anti-)symmetrization is irrelevant (we can still do it, but it makes no difference to the math)
Example: One e^- inside you and another on the moon (see example next page and also Shankar p.273) .

Example: Relevance of many-body symmetries:

Consider two particles with a Gaussian 1D wave-function, one centered at x_a the other at x_b , with width σ each. We want to compare distinguishable particles with non-symmetrized wavefunction $\psi_{\text{dist}}(x_1, x_2) = \varphi_a(x_1)\varphi_b(x_2) \equiv \mathcal{N}^2 \exp[-(x_1 - x_a)^2/(2\sigma^2)] \exp[-(x_2 - x_b)^2/(2\sigma^2)]$ with indistinguishable particles, where we (anti-)symmetrized the wave function $\psi_{\text{indist}}(x_1, x_2) = [\varphi_a(x_1)\varphi_b(x_2) \pm \varphi_b(x_1)\varphi_a(x_2)]/\sqrt{2}$. \mathcal{N} is a normalisation factor. Let us distinguish two cases:

case (i) $\sigma \ll |x_a - x_b|$, i.e. x_1 within you, x_2 on the moon.

Let us first draw the two wave-functions in a many-body coordinate system:



left: Non-symmetrised and symmetrised wave functions for case (i). (blue) $\psi > 0$, (red) $\psi < 0$, (white) $\psi \approx 0$.

Now consider the expectation value of any observable \hat{O} . For simplicity we assume \hat{O} contains no derivatives.

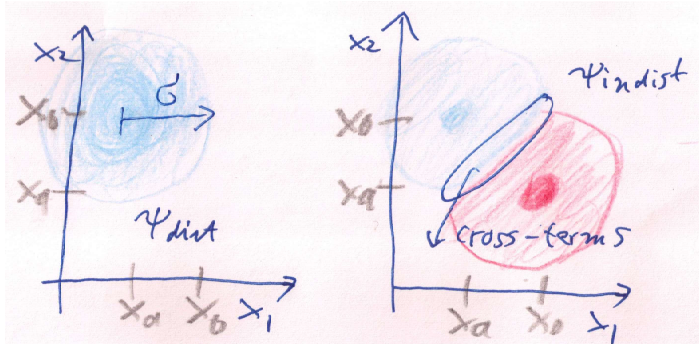
$$\langle \hat{O} \rangle = \int dx_1 dx_2 \psi^*(x_1, x_2) \hat{O} \psi(x_1, x_2)$$

$$\begin{cases} = \int dx_1 \int dx_2 |\varphi_a(x_1)|^2 |\varphi_b(x_2)|^2 \hat{O}, & \text{for } \psi_{\text{dist}} \\ = \int dx_1 \int dx_2 \left(|\varphi_a(x_1)|^2 |\varphi_b(x_2)|^2 \pm \underbrace{\frac{1}{2} [\varphi_a^*(x_1)\varphi_b(x_1)\varphi_b^*(x_2)\varphi_a(x_2)]}_{\approx 0} \right) \hat{O}, & \text{for } \psi_{\text{indist}} \end{cases} \quad (1.33)$$

- In the second line we used that we can rename $x_1 \leftrightarrow x_2$ in the integration.
- The term in square brackets vanishes because the overlap of the Gaussians near x_a and x_b is essentially zero.

We thus see that in this case, both approaches give the same result.

case (ii) $\sigma \sim |x_a - x_b|$, e.g. two electrons in an atom



left: Non-symmetrised and symmetrised wave functions for case (ii). Legend as above.

In this case the argument above no longer works, so the extra terms matter and can be crucial. In particular the overall integration for the expectation value depends on the sign \pm , so differs for Bosons and Fermions.