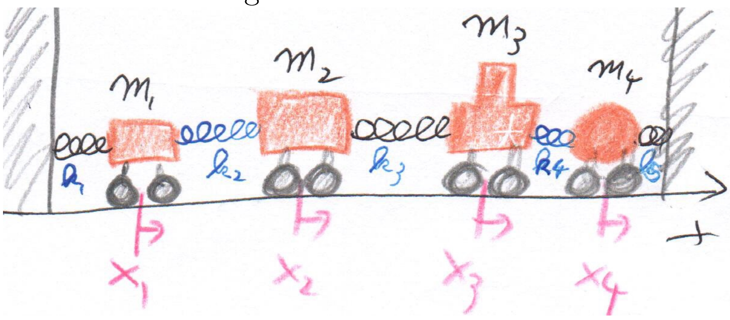
PHY 305, I-Semester 2020/21, Tutorial 7 solution

Stage 1 **Coupled oscillators** Extend the picture of section 3.6.1. to four coupled carts as shown in the figure below:



left: Four coupled carts.

(a) Write down the equations of motion.

Solution:

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1), \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) + k_3(x_3 - x_2), \\ m_3\ddot{x}_3 &= -k_3(x_3 - x_2) + k_4(x_4 - x_3), \\ m_4\ddot{x}_4 &= -k_5x_4 - k_4(x_4 - x_3), \end{aligned} \tag{1}$$

(b) Cast those in the form of a matrix equation as done in the lecture.

Solution:

$$\ddot{\mathbf{x}}(t) = \tilde{M}\mathbf{x}(t), \text{ with matrix } \tilde{M} = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_3+k_2}{m_2} & \frac{k_3}{m_2} & 0 \\ 0 & \frac{k_3}{m_3} & -\frac{k_4+k_3}{m_3} & \frac{k_4}{m_3} \\ 0 & 0 & \frac{k_4}{m_4} & -\frac{k_4+k_5}{m_4} \end{bmatrix}. \tag{2}$$

(c) Without doing the calculation, discuss in your team how you would solve the problem of finding the time-evolution of those four carts from a known initial condition.

Solution: We diagonalize the matrix obtained with the matrix multiplication $A = M^{-1}K$ *and find its eigenvectors and eigenvalues. We then construct the general solution according to the Ansatz (3.83) from the lecture. Finally, we evaluate* $\mathbf{q}(0)$ *and* $\dot{\mathbf{q}}(0)$ *from that Ansatz at time* $t = 0$ *and equate those to our given initial condition. This will give us an equation system for the coefficients* a_ℓ *which we can solve to find the* a_ℓ*.*

Stage 2 Hamiltonian mechanics

(a) What is the motivation to develop Hamiltonian mechanics? What is phase-space? Why is phase-space a helpful concept? What does Lioville's theorem say? Why is it useful?

Solution: The motivation to develop Hamiltonian mechanics is to put the two variables used, q and p on equal footing and to find equations of motion (EOM) that are first order in time, so that for any given choice of variables q, p there is a unique future evolution. Phase space is the space of all vectors $(\mathbf{q}, \mathbf{p})$, where we have combined coordinates and momenta into one single (possibly large) vector. It is helpful because we can often draw mechanical trajectories in it without solving the EOMs, e.g. by energy conservation arguments. Lioville's theorem says that the phase-space volume occupied by an ensemble of trajectories remains constant. This is a basic pillar of statistical mechanics, or thermodynamics.

(b) To see better why we need a Legendre transformation to go from a Lagrangian to a Hamiltonian, let's go back to the mathematical Legendre transformation on the level of a function, Eq. (4.3). For simplicity, we omit the second argument y. Consider two functions $f(x)$, for which we want to change variables to $u = \partial f/\partial x$: $f_1(x) = ax^2$ and $f_2(x) = a(x+b)^2$. For both of these, (a) find $u(x)$, (b) write a function $\tilde{f}(u)$ by solving your expression for $u(x)$ for x and eliminating x from $f_k(x)$ in favor of u. (c) Instead, find the Legendre transformations $g(u)$ of these two functions. Compare the results of the two approaches, what do you observe?

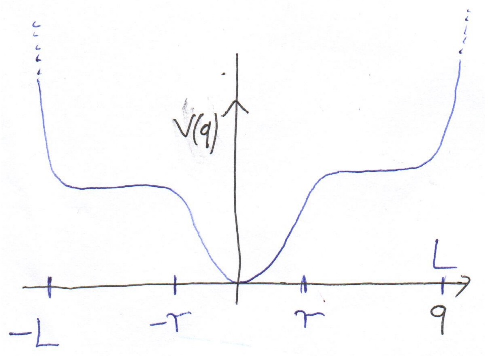
Solution: The derivative for test function f_1 is $u_1 = \partial f_1(x)/\partial x = 2ax$ and for f_2 we have $u_2 = \partial f_2(x)/\partial x = 2a(x+b)$. Replacing x by direct substitution would give us $\tilde{f}_1(u_1) = \frac{u_1^2}{4a}$ and $\tilde{f}_2(u_2) = \frac{u_2^2}{4a}$. If we do a Legendre transformation instead, we reach $g_1(u_1) = u_1 x - f_1(u_1) = \frac{u_1^2}{4a}$ versus $g_2(u_2) = u_2 x - f_2(u_2) = \frac{u_1^2}{4a} - u_2 b$. We see that when doing the direct substitution, we lost information encoded in the functional form of f_k (the value of b). This is avoided when using the Legendre transformation. We can even do the Legendre transformation a second time and get the original functions back (try it), proving that information is preserved.

(c) Now discuss why we need a Legendre transformation to go from a Lagrangian to a Hamiltonian.

Solution: Since either function is supposed to govern the dynamical evolution of our coordinates, we have to make sure that we do not loose any information in the conversion. You see in the example (b) that direct substitution does not guarantee this. Turns out, the Legendre transformation does.

Stage 3 Phase space

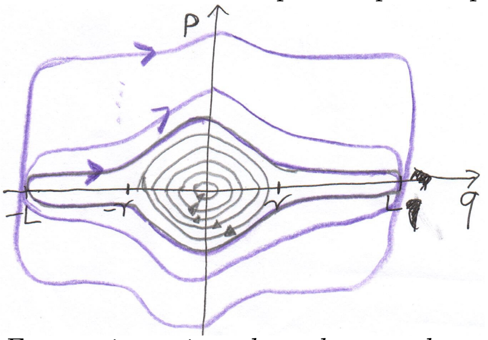
(a) Consider a particle in a 1D potential $V(q)$ drawn below, with Hamiltonian $H = \frac{p^2}{2m} + V(q)$. PTO



left: Drawing of potential

(b) Based on the drawing and your knowledge of basic physics, draw a qualitative phase-space portrait for that Hamiltonian.

Solution: *The phase-space portrait looks roughly as follows:*



left: *Phase-space portrait of dip-potential within box.*

For trajectories that do not have enough energy to leave the central, harmonic-oscillator shaped part between $-r$ and r, the phase space should look like that of the harmonic oscillator (Example 3). This is drawn as black lines. When the energy exceeds the plateau energy, trajectories can leave that central region and reach the plateau. While on the plateau, they experience no force since the potential is roughly constant there. Hence in that part the momentum stays constant, after moving with roughly constant velocity from r to q, the particle hits a wall from which it elastically reflects, turning the momentum from p to $-p$ within a very small spatial region. These trajectories are drawn in violet. All these features are reflected qualitatively in the drawn phase space portrait. Advanced Bonus: If you don't believe this, invent a mathematical form for $V(q)$ that looks as our drawing, insert into the numerical code given for Assignment 6 Q4, and numerically generate an exact portrait.