PHY 305, I-Semester 2020/21, Tutorial 5 solution

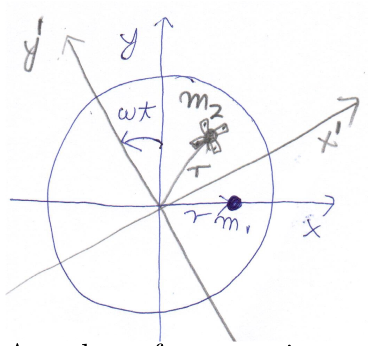
Stage 1 What is the difference between an inertial and a non-inertial frame? What is a fictitious force? Why do we need fictitious forces in non-inertial frames (answer both: mathematically and physically)?

Solution: Newtons equation in the form $\mathbf{F} = m\mathbf{a}$*, where* $\mathbf{F}$ *are all real forces only, holds only in an inertial frame. Real forces are those with a microscopic origin (molecular bonds, Coulomb force mediated by photons etc.). All inertial frames are at constant relative motion with respect to one another. Fictitious forces are those not fitting the description for real forces above: They arise only mathematically, if you insist to do physics in a non-inertial frame. However in your experience, they are indistinguishable from real forces. Mathematically fictitious forces pop up, since when you write* $\mathbf{x}'(t) = \mathbf{x}(t) + \mathbf{v}(t)$ *for nonconstant* $\mathbf{v}(t)$*, the time-derivatives in the acceleration* $\ddot{\mathbf{x}}'(t)$ *will also act on* $\mathbf{v}(t)$ *and cause extra terms. Physically they are required to translate the statement "a force free object in the frame at rest moves in a straight line" into the accelerated frame: There this cannot be true, hence we require additional forces acting in the non-inertial frame to make the trajectory deviate from a straight (+constant velocity) line in that frame.*

Stage 2 Describe in your own words how the concept of "moment of inertia" and "inertia tensor" arises. What are good and bad choices of coordinates for the calculations of moments of inertia?

Solution: When we fix a rotation velocity (+axis) and look at what happens to a rigid body (with all body elements at a fixed distance from one another), we see that after having fixed the rotation vector $\boldsymbol{\omega}$ *we can calculate the entire angular momentum and the rotational kinetic energy. Since we fixed only the rotation vector* $\boldsymbol{\omega}$*, we would like to express both quantities in terms of* $\boldsymbol{\omega}$*. The object linking them is the inertia tensor.*
A good choice for coordinate axes (and origin) are those for which the object has some symmetries. Choosing symmetry axes will typically make some products of inertia zero, or even all (such that the inertia tensor is directly diagonal).

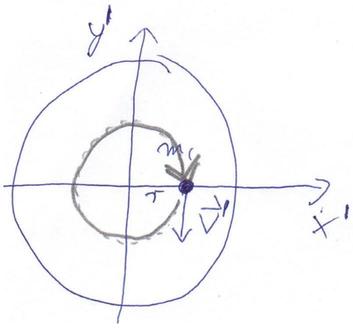
Stage 3 (i) *(Rotating frames I)*



left: Consider the blue mass m_1 which slides without friction on the turntable shown, a distance r from the centre of the table. The mass is at rest in the fixed frame x, y. The turntable rotates with angular velocity ω.

Are there forces acting on the mass in the frame that rotates with the turntable? If yes, find them all....

Solution: Since the mass is at rest in the lab-frame, the real forces $\mathbf{F}$ *in Eq. (2.90) of the lecture are zero. But the fictitious forces are not, since they are needed to make the mass move in a circle in the rotating frame, which it logically has to move on. We can think of the lab-frame as rotating with an angular velocity* $-\boldsymbol{\omega}$ *with respect to the turntable frame, hence using Eq. (3.12) the velocity of mass* 1 *in the turntable frame is* $\boldsymbol{v}' = -\boldsymbol{\omega} \times \boldsymbol{r}' = \boldsymbol{r} \times \boldsymbol{\omega}$. *The velocity is pointed in downward direction as shown in the figure. The centrifugal force acting on the mass is* $\boldsymbol{F}_{cf} = -m(-\boldsymbol{\omega}) \times (-\boldsymbol{\omega} \times \boldsymbol{r}) = m\omega^2 r$ *(outward), and the coriolis force is* $\boldsymbol{F}_{cor} = -2m(\boldsymbol{v}') \times -\boldsymbol{\omega} = 2m\omega^2 r$ *(inward). Therefore, in the rotating frame there is a net force of* $\boldsymbol{F}_{net} = \boldsymbol{F}_{cor} - \boldsymbol{F}_{cf} = m\omega^2 r$ *(inward) to account for the centripetal acceleration.*



left: Dynamics of the blue mass m_1 which slides without friction on the turntable shown in frame $x'y'$ at a distance r from the centre of the table. The velocity vector $\boldsymbol{v}$ is pointed in downward direction.

(ii) *(Rotating frames II)* Now consider the second mass m_2 instead, which is initially bolted to the turntable so it is fixed in the frame x', y'. While it is bolted, what are the forces acting on the bolts? *Solution: Since the mass* m_2 *is fixed in the rotating frame, it does not move there* $\mathbf{v}' = 0$. *Hence the Coriolis force is zero. However there will be a centrifugal force* $\boldsymbol{F}_{cf} = -m(-\boldsymbol{\omega}) \times (-\boldsymbol{\omega} \times \boldsymbol{r}) = m\omega^2 r$ *(outward) just as before, which must be compensated by the bolts in order for the mass to remain at rest in the rotating frame.*

(iii) *(Rotating frames III)* If the mass is released from the bolts, it travels in a straight line in the space fixed frame (x,y). What path does it take in the rotating frame of the turntable (x',y')? *Solution: See Morin Page IX-17, Q5.*

Stage 4 Moments of inertia, cross-products

(i) Find the moment of inertia tensor for the following collections of four masses m_1, m_2, m_3, m_4, for rotations around the $z-axis$. Start with making a drawing of the configuration. Relate this to your understanding of angular momentum and rotational kinetic energy for such a rotation.

- $m_1 = 1$ kg, $\mathbf{r}_1 = [1, 0, 0]^T$ m, $m_2 = 1$ kg, $\mathbf{r}_1 = [-1, 0, 0]^T$ m, $m_3 = 5$ kg, $\mathbf{r}_1 = [0, 0, 0]^T$ m, $m_4 = 5$ kg, $\mathbf{r}_1 = [0, 0, -3]^T$ m.
- $m_1 = 1$ kg, $\mathbf{r}_1 = [2, 0, 0]^T$ m, $m_2 = 1$ kg, $\mathbf{r}_1 = [-2, 0, 0]^T$ m, $m_3 = 5$ kg, $\mathbf{r}_1 = [0, 0, 0]^T$ m, $m_4 = 5$ kg, $\mathbf{r}_1 = [0, 0, -3]^T$ m.
- $m_1 = 1$ kg, $\mathbf{r}_1 = [1, 0, -1]^T$ m, $m_2 = 1$ kg, $\mathbf{r}_1 = [-1, 0, 1]^T$ m, $m_3 = 5$ kg, $\mathbf{r}_1 = [0, 0, 0]^T$ m, $m_4 = 5$ kg, $\mathbf{r}_1 = [0, 0, -3]^T$ m.

- $m_1 = 2$ kg, $\mathbf{r}_1 = [1, 0, -1]^T$ m, $m_2 = 2$ kg, $\mathbf{r}_1 = [-1, 0, 1]^T$ m, $m_3 = 2$ kg, $\mathbf{r}_1 = [1, 0, 1]^T$ m, $m_4 = 2$ kg, $\mathbf{r}_1 = [-1, 0, -1]^T$ m.

Solution:

- *Let us assume that the coordinate axes are placed at one of the corner of a cube as shown in the Fig 1. If we place m_3 at the origin of the coordinate axes and other masses as indicated in the figure, the moment of inertias can be found as listed below (we skipped writing units. Masses are in kg and positions in m, so the moment of inertia is in kgm^2.*

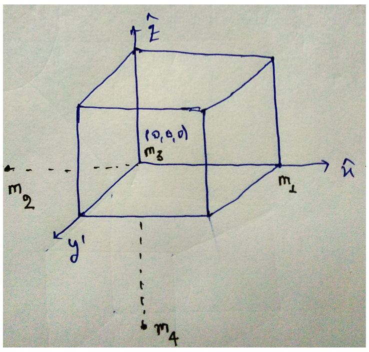


Figure 1: The coordinate axes are shown here for first collection of masses. Origin is placed at one of the corner of the cube as $(0, 0, 0)$.

$$\begin{aligned} I &= \sum_i m_i r_i^2 \\ &= 1(1)^2 + 1(-1)^2 + 5(0)^2 + 5(0)^2 \\ &= 2. \end{aligned} \tag{1}$$

- *The moment of inertia assuming similar senario as for the above:*

$$\begin{aligned} I = I &= \sum_i m_i r_i^2 \\ &= 1(2)^2 + 1(-2)^2 + 5(0)^2 + 5(0)^2 \\ &= 8. \end{aligned} \tag{2}$$

- *Again we have similar situation as for the above:*

$$\begin{aligned} I &= 1(1^2 + -(1)^2) + 1(1^2 + (-1)^2) + 5(0)^2 + 5(0)^2 \\ &= 4. \end{aligned} \tag{3}$$

- *The coordinate axes can again be taken similar as for the above cases but this time mass m_3 is shifted from corner to the given coordinate. The moment of inertia reads as:*

$$\begin{aligned} I &= 2(1^2 + (-1)^2) + 2(1^2 + (-1)^2) + 2(1^2 + 1^2) + 2((-1)^2 + (-1)^2) \\ &= 16. \end{aligned} \tag{4}$$

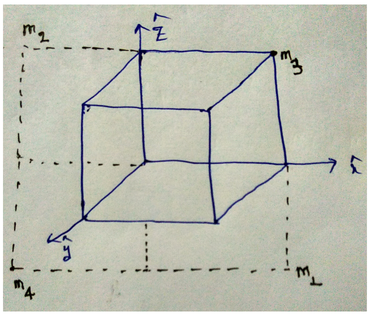


Figure 2: The coordinate axes are shown here for last collection of masses. Origin is placed at one of the corner of the cube as linked with z-axis.

It is to noted here that the masses are rotating about the z-axis, so other components of inertial tensor do not contribute to it.

(ii) Using the epsilon-tensor and the rule $\sum_i \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, show Eq. (2.51) and Eq. (3.23) of the lecture:

$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}), \\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \end{aligned} \tag{5}$$

for 3-component vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
Solution: Writing in component notation, we have

$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &\overset{Eq.\ (2.96)}{=} \sum_n a_n(\sum_{jk} \epsilon_{njk} b_j c_k \\ &= \sum_{njk} \epsilon_{njk} a_n (b_j c_k) = \sum_{njk} \epsilon_{jkn} b_j (c_k a_n) = \sum_{njk} \epsilon_{knj} c_k (a_n b_j). \end{aligned} \tag{6}$$

The last two equalities are due to the possibility to cyclicly change the indices of the epsilon tensor, and when translating them back to scalar and vector products, these two give us what was to be shown. Similarly

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c})\Big|_n &\overset{Eq.\ (2.96)}{=} \sum_{jk} \epsilon_{njk} a_j (\mathbf{b}\times\mathbf{c})\Big|_k = \sum_{jk} \epsilon_{njk} a_j \left(\sum_{\ell m} \epsilon_{k\ell m} b_\ell c_m\right) \\ &= \sum_{jk\ell m} \epsilon_{njk}\epsilon_{k\ell m} a_j b_\ell c_m = \sum_{j\ell m} \underbrace{\sum_k \epsilon_{knj}\epsilon_{k\ell m}}_{=\delta_{n\ell}\delta_{jm} - \delta_{nm}\delta_{j\ell}} \ a_j b_\ell c_m \\ &= \sum_j a_j b_n c_j - \sum_{jk} a_j b_j c_n = \mathbf{b}\Big|_n (\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\Big|_n (\mathbf{a}\cdot\mathbf{b}). \end{aligned} \tag{7}$$

We used $\mathbf{v}\Big|_n$ *for the component* v_n *of vector* $\mathbf{v}$.