PHY 305, I-Semester 2020/21, Tutorial 3 solution

Stage 1 *(Variational techniques)* Discuss:

(i) What is a functional? Try to invent some new functionals with physical meaning that were not examples in the lecture.

Solution: A functional is a map from one or many functions onto a number. You can think of it as a "function of functions". Some physics problems that can be expressed as a functional are e.g. (i) the amount of heat dissipated during motion of a rocket through the atmosphere (integral over heating rate as a function of time, which will depend on the path taken), (ii) the time it takes for a boat to cross a river between two given points depending on the path taken on the river, (iii) The total power dissipated in an electric circuit, dependent on certain externally controllable applied voltage profiles $V(t)$.

(ii) Explain the problem statement of finding an extremal point of a functional using variational techniques.
Solution: When we want to find a stationary point of a functional, it means an input function $y_0(x)$ so that for <u>*all possible small deviations*</u> *from this function, independently at all positions x, the first order change of the value of the functional is zero.*

(iii) Identify the critical tricks in practically solving the problem.
Solution: There are at least two tricks: First we have reduced the problem of finding the stationary point of the functional to finding a minimum of a function of a single parameter α, by expression the deviation from the real path as $y(x) = y_0(x) + \alpha\eta(x)$. Secondly, we need to use the fact that $\alpha = 0$ ought to be a minimum of the functional <u>*for all*</u> *possible functions $\eta(x)$. For that reason we can set the function that multiplies $\eta(x)$ in the integrand to zero* <u>*at all values of x. See lecture eqns. (2.9)-(2.11) for all of this.*</u>.

Stage 2 *(Lagrangian mechanics)* Discuss:

(i) How come we suddenly have a completely new formalism for mechanics?
Solution: The new formalism (Lagrange) is based on exactly the same starting points as the old one, so it neither invalidates nor extends the old one (Newton). But it uses more advanced mathematical concepts, and thus has the advantages listed in the next point.

(ii) List the primary strengths of Lagrangian mechanics. In which cases or why is it more powerful than Newtonian? *Solution: (i) The Lagrangian formalism is "covariant", which means the equation of motion has exactly the same mathematical form in all coordinate systems, which makes it* *easier to use in non-cartesian coordinates. (ii) It efficiently deals with constraints, essentially one does not have to worry about constraint forces at all. (iii) Lagrange allows natural predictions of conserved quantities and analyses of their origin [week 4].*

(iii) What is the procedure of solving a problem in the Lagrangian approach? Where do you have to be careful?
Solution: (i) We first setup the Lagrangian, which is typically most straightforward in the original (unconstrained) coordinates. We have to be careful that the reference frame chosen is inertial, (ii) We then convert that Lagrangian into the new generalised coordinates using the coordinate transformation equation chosen and differential calculus. (iii) We then derive the Lagrange equations from the Lagrangian, carefully treating all generalised coordinates and their generalise velocities as independent when taking partial derivatives.

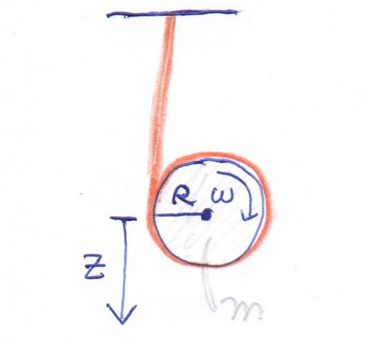

Figure 1: Sketch of yoyo. A string (brown) is held with your finger and wound around a cylinder (grey) of mass m with radius R. The yoyo can move down by unwinding the string while rotating, and up by winding up the string through rotation.

Stage 3 *(Applications)* Solve whichever you want first:

(i) Figure 1 shows a sketch of a "yoyo" toy, see caption. Discuss the constraints, write down the Lagrangian in some suitable generalized coordinate and derive the Euler Lagrange equations.
Solution: The yo-yo will have both rotational and translational kinetic motion. Hence we can write it's kinetic energy as,

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2, \tag{1}$$

where $\dot{x}$ is the translational velocity v and $\dot{\phi}$ is the angular velocity ω. The moment of inertia for a yoyo (taking it to be a solid disk) is given by $I = \frac{1}{2}mR^2$. The rope does not slip as the yo-yo falls which brings the constraint $\dot{x} = \dot{\phi}R$. Altogether we have:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{\dot{x}}{R}\right)^2 = \frac{3}{4}m\dot{x}^2. \tag{2}$$

The potential energy will be:

$$U = -mgx. \tag{3}$$

Hence we can write the Lagrangian of the system to be:

$$L = \frac{3}{4}m\dot{x}^2 + mgx. \tag{4}$$

The Euler-Lagrangian equation will be,

$$\frac{\partial L}{\partial x} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{5}$$

$$\Rightarrow mg - \frac{d}{dx}\left(\frac{3}{2}m\dot{x}\right) = \ddot{x} = \frac{2}{3}g \tag{6}$$

(ii) Find a function $y(x)$ that makes the functional

$$S[y] = \int_0^1 \sqrt{1 + x + y'(x)^2}dx \tag{7}$$

with $y(0) = 0$, $y(1) = 1$ stationary.
Solution: We can write Euler-Lagrangian equation for the function as,

$$-\frac{d}{dx}\left[\frac{\partial}{\partial y'}\left(\sqrt{1 + x + y'(x)^2}\right)\right] = 0, \tag{8}$$

$$\Rightarrow \frac{\partial}{\partial y'}\left(\sqrt{1 + x + y'(x)^2}\right) = const. = C. \tag{9}$$

This gives,

$$y' = C\left(\sqrt{1 + x + (y')^2}\right). \tag{10}$$

This can be rearranged to rewrite y' in terms of x of the form,

$$C_1\, y' = \sqrt{1 + x}, \tag{11}$$

where C_1 is a new constant after rearrangement. If we integrate wrt x we get,

$$C_1\, y = \frac{2}{3}\,(1 + x)^{3/2} + C_2. \tag{12}$$

At this stage we can use the conditions $y(0) = 0$, $y(1) = 1$ to find the constants C_1 and C_2.

$$y(0) = 0 \Rightarrow\ C_2 = -\frac{2}{3} \tag{13}$$

$$y(1) = 1 \Rightarrow\ C_1 = \frac{2}{3}\left[(2)^{3/2} - 1\right] \tag{14}$$

Hence we can write the final equation to be,

$$y = \frac{(1 + x)^{3/2} - 1}{2^{3/2} - 1} \tag{15}$$