PHY 305, I-Semester 2020/21, Tutorial 2 Solution

Work in the same teams as for assignments. Do "Stages" in the order below. Discuss via online (video or audio) conference on a subchannel for your group.

Stage 1 *(math review)* Review from old course notes, books or the internet your knowledge about (ordinary, linear) differential equations. Make sure you are comfortable with the answer to all these questions:

(i) What does an equation like $\dot{f}(t) = \frac{d}{dt}f(t) = g(t)$ tell me? What mathematical object is the solution of such an equation? What mathematical operation is in general required to solve such an equation and why?
Solution: The equation states that at every time t (if t is time, it does not have to be), the slope of the function $f(t)$ will be $g(t)$. The solution of such an equation is a function $f(t)$. Since integration is the inverse operation to differentiation, we typically need an integration to solve such a differential equation.

(ii) For a differential equation $\frac{d^n}{dt^n}f(t) = g(t)$, how many unspecified constants does the solution contain and why? Which additional information can help to constrain those constants?
Solution: The equation is of n'th order in time and thus will contain n unspecified constants. The reason is, that for every integration $\int h(t)dt$ we get $\int h(t)dt = H(t) + C$, where C can be any constant.

Stage 2 *(physics review)* Review from old course notes, books or the internet your knowledge about fundamental laws of physics. Make sure you are comfortable with the answer to all these questions:

(i) A cycle is moving on a street with constant velocity such that its position in a street fixed reference frame is $x(t) = x_0 + v_0 t$. You are observing it from a train going parallel with velocity $v_1 \gg v_0$. What is the position in that cycle in your reference frame, which we assume is fixed on your seat in the train?
Solution: The positions is $x'(t) = x'_0 + (v_0 - v_1)t$, where we don't know x'_0 because I did not tell you where the seat is.

(ii) Why can the zero of energy be chosen arbitrarily?
Solution: Because energy contains the potential energy which enters the physics only via forces, which in turn are defined via a derivative. Any constant offset drops out in that derivative.

(iii) Convert the following 2D cartesian coordinates into polar ones: $(x, y) = (0, 2), (2, 0), (1, 1), (-3, -3), (-2, 0), (0, 0)$.
Solution: $(r, \varphi) = (2, \pi/2),\ (2, 0),\ (\sqrt{2}, \pi/4),\ (3\sqrt{2}, -3\pi/4),\ (2, \pi),\ (0, \text{undefined})$.

(iv) Convert the following 2D polar coordinates into cartesian ones: $(r, \varphi) = (4, \pi/4), (2, \pi), (3, 0), (3, 8\pi), (2, 3\pi/2), (0, \pi/8)$.
Solution: $(x, y) = (2\sqrt{2}, 2\sqrt{2}),\ (-2, 0),\ (3, 0),\ (3, 0),\ (0, -2),\ (0, 0)$

Stage 3 (i) Think like a computer: How would you go about solving the differential equation $\dot{f}(t) = \frac{d}{dt}f(t) = g(t)$ by dividing time into small chunks of size Δt, thus allowing only discrete times t_0, $t_1 = t_0 + \Delta t$, $t_2 = t_0 + 2\Delta t$ etc...? Make a sketch of this computational solution versus the real solution.

Solution: We can go back to the definition of a derivative which can be e.g. $\dot{f}(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$. *Now, since the computer cannot do* $\lim_{\Delta t \to 0}$ *we just take that definition at some small but finite* Δt*, called the "step size". Inserting into the ODE gives*

$$\begin{aligned} \dot{f}(t) &= \frac{d}{dt}f(t) = g(t), \\ \frac{f(t+\Delta t) - f(t)}{\Delta t} &= g(t), \\ f(t+\Delta t) &= f(t) + \Delta t \;\, g(t). \end{aligned} \tag{1}$$

The last line tells the computer how to find $f(t+\Delta t)$ *if it knows* $f(t)$. *Since that equation is linear in* t*, the real function will be approximated by lots of short straight line segments as shown in the sketch. Sketch see Fig. 1.*

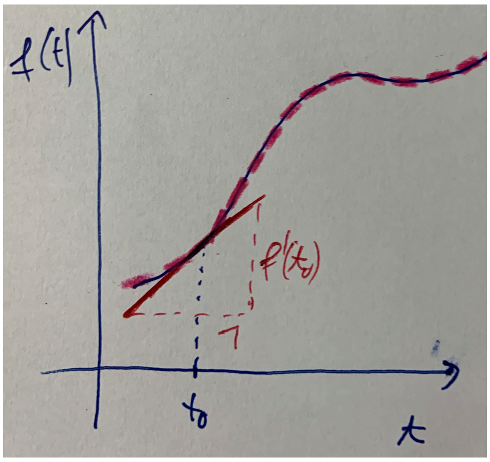

Figure 1: Approximation of ODE solution through lots of short straight line segments.

(ii) You are throwing a stone very far, subject to gravity and friction (but not wind). How many dimensions does the relevant phase space have? Make a sketch of the stone's trajectory in the (x, z), (x, p_x) and (z, p_z) phase space planes (gravity along z).

Solution: Since we can neglect sideways motion and chose our x*-axis in the direction the stone was thrown, 4 dimensions are enough* (x, z, p_z, p_z). *For sketches see Fig. 2.*

(iii) Show that for a conservative potential, such that $\mathbf{F} = -\nabla V(\mathbf{r})$ we have $W_{12} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$.

Solution: Inserting the definition of the line-integral for the expression of

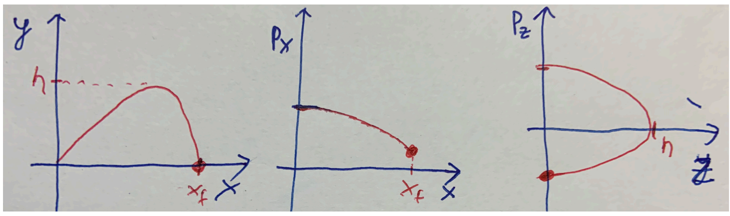

Figure 2: Stone trajectories on cuts through 4D phase space.

work we have

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\begin{aligned}
W_{12} &= \int_1^2 \big[-\boldsymbol{\nabla} V(\mathbf{s}(t)) \big] \cdot \frac{\partial}{\partial t}\mathbf{s}(t) dt \\
&= -\int_1^2 \frac{d}{dt} V(\mathbf{s}(t)) dt = -[V(\mathbf{s}(2)) - V(\mathbf{s}(1))] = V(\mathbf{r}_1) - V(\mathbf{r}_2). \quad (2)
\end{aligned}
$$

The trick was in the second step, which we write again backwards in full detail expanding in vector components:

$$
\begin{aligned}
\frac{d}{dt} V(\mathbf{s}(t)) &= \frac{d}{dt} V(s_x(t), s_t(t), s_z(t)) \\
&= \frac{\partial}{\partial x} V(s_x(t), s_t(t), s_z(t)) \frac{\partial}{\partial t} s_x(t) + \frac{\partial}{\partial y} V(s_x(t), s_t(t), s_z(t)) \frac{\partial}{\partial t} s_y(t) \\
&\quad + \frac{\partial}{\partial z} V(s_x(t), s_t(t), s_z(t)) \frac{\partial}{\partial t} s_z(t) \\
&= \boldsymbol{\nabla} V(\mathbf{s}(t)) \cdot \frac{\partial}{\partial t} \mathbf{s}(t). \quad (3)
\end{aligned}
$$