PHY 305, I-Semester 2020/21, Tutorial 1 Solution

Stage 1 *(math review)* Review your knowledge about vectors from old course notes, books or the internet. Make sure you are comfortable with the answer to all these questions:

(i) What is a vector? How do you find the length of a vector? How do you find the angle between two vectors? When are two vectors orthogonal?
Solution: Mathematically, a vector is an element of a vector space (see below). In physics we can mostly think of them as groups of numbers such as $\mathbf{v} = [v_1,\ v_2,\ v_3]^T$*, that you can then combine with useful operations, such as addition or multiplication with scalars. For that vector the length is* $|\mathbf{v}| \equiv \sqrt{v_1^2 + v_2^2 + v_3^2}$*. The angle* α *between two vectors* $\mathbf{v}$, $\mathbf{w}$ *is given by* $\cos(\alpha) = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$*, where* $\cdot$ *is a scalar product. Thus they are orthogonal when* $\mathbf{v}\cdot\mathbf{w} = 0$.

(ii) What is a vector space? How do you change basis in a vector space?
Solution: A vector space is a collection of vectors together with certain operations among them (addition and multiplication by scalar), that satisfy a list of rules (see math book). For practical purposes here, we will most commonly find a vector space such as $\mathbb{R}^3$ *with means the "collections of all possible* $\mathbf{v} = [v_1,\ v_2,\ v_3]^T$ *with three real numbers* v_k*". Similarly* $\mathbb{R}^d$ *for other numbers of dimensions. An orthonormal basis of a d-dimensional vector space is a set of d orthogonal unit vectors* $\mathbf{b}_k$ *which allow you to write ANY vector* $\mathbf{v}$ *in that space as*

$$\mathbf{v} = \sum_k c_k \mathbf{b}_k, \tag{1}$$

For the right set of coefficients c_k*. If you want to write this in another basis* $\mathbf{v} = \sum_k \tilde{c}_k \tilde{\mathbf{b}}_k$*, you can use the scalar product and conditions for orthogonality to see* $\tilde{c}_k = \sum_n c_n \mathbf{b}_n \cdot \tilde{\mathbf{b}}_k$*.*

(iii) How do you add or substract vectors, multiply them with a scalar, or multiply two vectors? What is a scalar-product or a cross-product?
Solution: See mathbook for the first three operations. There are two different main ways to multiply two vectors: The outcome of the <u>*scalar product*</u> *is a scalar (i.e. a number, not a vector), as in* $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$*. The outcome of the* <u>*cross product*</u> *(also "vector product") is itself a vector again,* $\mathbf{v}\times\mathbf{w} = (v_2w_3 - v_3w_2)\hat{\mathbf{e}}_x + (v_3w_1 - v_1w_3)\hat{\mathbf{e}}_y + (v_1w_2 - v_2w_1)\hat{\mathbf{e}}_z$*.*

(iv) For two 3D vectors $\mathbf{v}_1$, $\mathbf{v}_2$ with different directions, show that $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. How can you use this information to mathematically define "a plane" (i.e. a straight 2D surface embedded in 3D)?
Solution: You can show the first statement by using the definition of the cross product for $\mathbf{v}_3$ *from item (iii), and then showing* $\mathbf{v}_1\cdot\mathbf{v}_3 = 0$ *and* $\mathbf{v}_2\cdot\mathbf{v}_3 = 0$*. A plane is spanned by two basis vectors* $\mathbf{b}_1$ *and* $\mathbf{b}_2$*. Using*

the rule above we know that $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$ is orthogonal to both and hence orthogonal to the plane. Now we can flip this around, and define the plane as the set of all vectors $\mathbf{v}$ that fulfill $\mathbf{n} \cdot \mathbf{v} = d$, where d is the distance of the plane from the origin in the direction of $\mathbf{n}$.

Stage 2 *(physics review)* Review your knowledge about fundamental laws of physics from old course notes, books or the internet. Make sure you are comfortable with the answer to all these questions:

(i) List three basic physical conservation laws. What are the conserved quantities? Under which conditions are they conserved?
Solution: Most common are the conservation of energy, momentum and angular momentum. Note the latter two really are three conservation laws each, since the quantity concerned is a vector. For energy to be conserved, the potential has to be time independent. For momentum to be conserved, there should not be an external force. For angular momentum to be conserved, there should not be an external torque. We will link these three statements with fundamental symmetry principles in week 4.

(ii) When are these laws useful/ how can you use them?
Solution: While we can in principle always find the dynamics of a system from Newton's equations, it is often much easier to use these conservations laws, i.e. finding the initial energy at $t = 0$, and then evaluating the position or velocity of a particle at a later time using energy conservation.

(iii) What is the basic equation of motion of mechanics? What do you need to specify to know the motion of a particle?
Solution: Newton's equation. It is a second order differential equation in time, hence to fullly specify $\mathbf{x}(t)$ we need to know the initial position <u>and</u> initial velocity.

(iv) A blob is sliding on the wobbly plane shown below (second page) without friction and was initially at rest and at the top as shown. Assume the wobbly plane is defined by the equation $y(x) = (H - h)\exp(-x/L) + h\cos(x/\lambda)$, find the x an y components of the velocity of the blob, once it has reached the x-coordinate x_f, assuming the only relevant force is gravity, with potential energy $V = mgy$.
Solution: Without the math: We can use energy conservation to find the kinetic energy at any coordinate x_f using $T(x_f) = V(x = 0) - V(x_f)$, where V is the potential energy due to gravity. Then we can find the tangential unit vector $t = y'/|y'|$ to the curve at the position x_f and thus split the velocity into its x and y components, since we know it has to be tangential to the wobbly plane.

Stage 3 Suppose you are in a stone-throwing competition on a weird planet. The stone has a mass m and you can throw it so hard that its initial velocity is v_0. At which angle to the surface of the planet do you have to throw it, such that it reaches farthest? We assume the only two forces acting on the stone after it

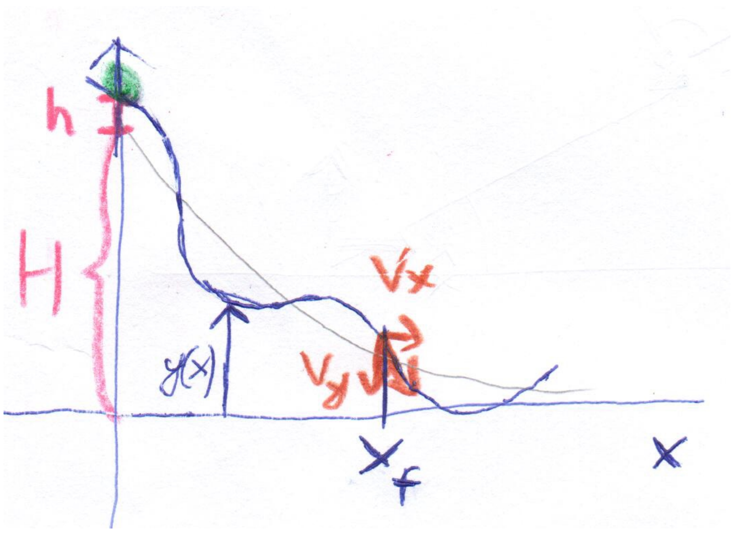

Figure 1: Sketch of wobbly plane (blue line), blob (green) and velocity (brown).

leaves your hand is gravity $\mathbf{F} = -mg\hat{\mathbf{z}}$ and linear drag[1] $\mathbf{F} = -\gamma\mathbf{v}$.
Solution: FOLLOWING SHORTLY. TA: see Taylor book for differential equations for x and y motion (2.15), (2.16). Solve y equation for $y = 0$ to find time $t > 0$ where stone returned to ground. Insert into $x(t)$ solution. Then differentiate with respect to angle to find maximum. If you know of a better solution, ignore these statements.

[1]This is why the planet is weird. On earth, typically quadratic drag would dominate.