PHY 305, I-Semester 2020/21, Assignment 6

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Due-date: 21.11.2020

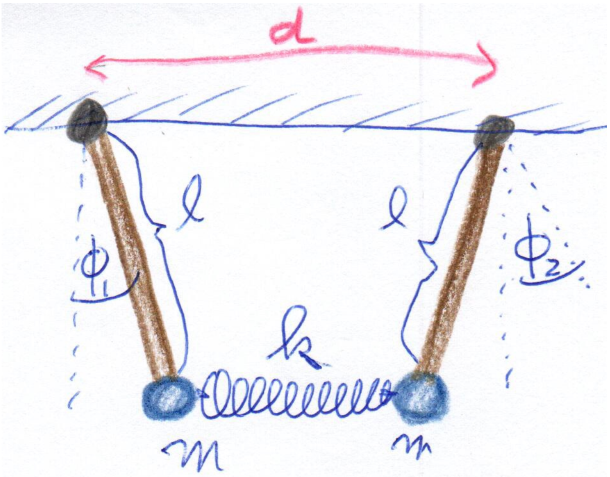

Figure 1: Coupled rigid pendulums as discussed in the text.

(1) Coupled pendulums: [**10pts**] Consider the two rigid pendulums shown in Fig. 1. For each, a bob of mass m is stuck to the end of a massless rigid rod of length ℓ. Their support is a distance d apart, and they are connected by a spring, of equilibrium length L and spring constant k.

(1a) Write the exact Lagrangian, using parameters and coordinates as in Fig. 1. Set up the equations to determine the equilibrium position of the system, without solving them. Instead of solving them, discuss your expectation how the equilibrium positions should change as parameters are varied.

(1b) Now for the special case $L = d$, find the equilibrium position with common sense, verify that it solves the equations found in (1a) and expand the Lagrangian to second order in coordinates as in the steps preceding Eqn. (3.78) in the lecture.

(1c) Find the normal modes and normal frequencies of the system in (1b).

(2) Pendulum cart: [**5pts**] Since we are now huge fans of carts and pendulums, let's stick a pendulum onto a cart. As usual spring and rod are massless, other variables are indicated in Fig. 2.

(2a) Write the exact Lagrangian, using parameters and coordinates as in Fig. 2, then convert it to a Hamiltonian using the Legendre transformation. Is your Hamiltonian equal to the total energy?

(2b) Setup Hamilton's equations, and expand those in terms of small angles ϕ.

(2c) [Bonus question] Using the techniques for solutions of coupled ODES in week 9, solve the equations. If need be using a computer, plot or draw the dynamics starting from $x_1 = \dot{x}_1 = 0$, $\phi(0) = 0.2\pi$ and $\dot{\phi} = 0$.

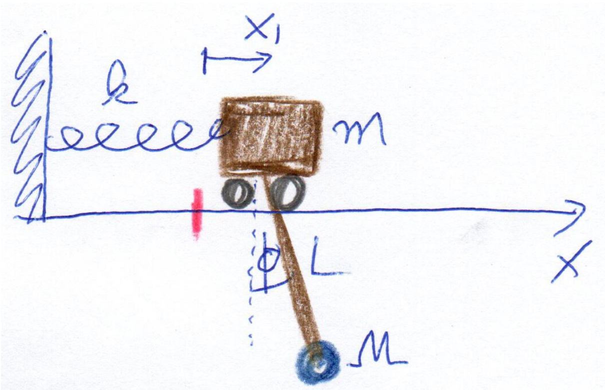

Figure 2: Simple pendulum mounted on a cart.

(3) Connected masses again: [10pts] Consider Q1 of Assignment 3 in the Hamiltonian formalism. In that question one mass was sliding on a table, connected to a second with a rope through a hole, see Assignment 3 sheet.

(3a) Describe the explicit conversion from the Lagrangian to the Hamiltonian function using the Legendre transform.

(3b) Setup Hamilton's equations, and show that they are equivalent to the Lagrange equations found earlier.

(4) Phase space of the simple pendulum: [10pts]

(4a) Let us make a more professional version of the bottom figure in example 41. Edit the equation of motion in `Assignment6_program_draft_v1.m` (which is based on the code of Assignment 1), such that the code solves the equations of motion of the simple pendulum (Example 41). It is already set up so that it solves the equation a few times, starting from different energies E. Use `Assignment6_oscillation_slideshow_v1.m` to get an idea of the pendulum motion for different E, then `Assignment6_phasespace_portrait_v1.m` to make the figure. For this you have to choose the right range and number of initial energies so the plot looks like the one from the lecture. Rename the final data from this part into `assignment6_output_parta.mat` for use in part (b).

(4b) We can also use that code to numerically verify Liouville's theorem. For that, change the initial state generation such that it randomly allocates initial positions and momenta within a rectangular phase space region centered on p_0, ϕ_0 with size Δp and $\Delta\phi$ (you just have to change `type`). You can reduce the final time to where you would expect 2-3 oscillations. Convince yourself (just by eye) with `Assignment6_swarm_evolution_v1.m` that the volume of this swarm of trajectories does indeed not change.