PHY 305, I-Semester 2020/21, Assignment 5

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Due-date: 24.10.2020

(1) Weighted dice: [**10pts**] Consider a game dice made with the intention of cheating, such that it is heavier on the bottom. We place the origin $O = (0, 0, 0)$ on one corner of the cube, such that the cube extends from $0 < x < a$, $0 < y < a$ and $0 < z < a$. The density shall be $\rho(\mathbf{r}) = \rho_0 \left(1 - \frac{z}{2a}\right)$, reflecting that the dice is not regular.

(1a) Find all components of the inertia tensor with respect to above coordinates and origin.

solution: The cube in the given problem is inhomogeneously distributed with a density $\rho(\mathbf{r}) = \rho_0 \left(1 - \frac{z}{2a}\right)$, *therefor we first calculate the total mass of the cube to express the inertial tensor in a more practical way. The total mass of the cube can be calculated as:*

$$\begin{aligned} M &= \int dm, \\ &= \int \rho(\vec{r}) dx dy dz, \\ &= \int_0^a dx \int_0^a dy \int_0^a \rho_0(1 - \frac{z}{2a}) dz, \\ &= \frac{3}{4} a^3 \rho_0, \end{aligned} \tag{1}$$

where $dm = \rho dV$ is the mass of a small element of the cube. Since we place the origin of coordinate axis at one of corner of the cube, the integration limit runs from 0 to a.

The different components of inertia tensor are:

$$\begin{aligned} I_{xx} &= \int dm(y^2 + z^2), \\ &= \int \rho(r) dx dy dz (y^2 + z^2), \\ &= \int \rho_0(1 - \frac{z}{2a})(y^2 + z^2) dx dy dz, \\ &= \frac{11}{18} M a^2. \end{aligned} \tag{2}$$

$$\begin{aligned} I_{xy} &= -\int dm xy, \\ &= -\int \rho(\vec{r}) xy dx dy dz, \\ &= -\frac{1}{4} M a^2. \end{aligned} \tag{3}$$

$$
\begin{aligned}
I_{xz} &= -\int dmxz, \\
&= -\int \rho(\vec{r})xzdxdydz, \\
&= -\frac{2}{9}Ma^2.
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
I_{yy} &= \int dm(x^2+z^2), \\
&= \int \rho(r)dxdydz(x^2+z^2), \\
&= \frac{11}{18}Ma^2.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
I_{yx} &= -\int dmyx, \\
&= -\int \rho(\vec{r})yxdxdydz, \\
&= -\frac{1}{4}Ma^2.
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
I_{yz} &= -\int dmyz, \\
&= -\int \rho(\vec{r})yzdxdydz, \\
&= -\frac{2}{9}Ma^2.
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
I_{zz} &= \int dm(x^2+y^2), \\
&= \int \rho(r)dxdydz(x^2+y^2), \\
&= \frac{2}{3}Ma^2.
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
I_{zx} &= -\int dmzx, \\
&= -\int \rho(\vec{r})zxdxdydz, \\
&= -\frac{2}{9}Ma^2.
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
I_{zy} &= -\int dmzy, \\
&= -\int \rho(\vec{r})zydxdydz, \\
&= -\frac{2}{9}Ma^2.
\end{aligned} \tag{10}
$$

The inertia tensor matrix can now be written as:

$$
I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}
$$

$$
I = \begin{bmatrix} \frac{11}{18}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{2}{9}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{11}{18}Ma^2 & -\frac{2}{9}Ma^2 \\ -\frac{2}{9}Ma^2 & -\frac{2}{9}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix}
$$

(1b) Diagonalize the inertia tensor and find the moments of inertia about the principal axes and the principal axes *(You may use e.g. Mathematica)*

solution: The diagonalization of the inertia matrix can be done following the usual matrix method in linear algebra, or one can use mathematica. Using mathematica, the moment of inertia along the principal axes and principal axes are:

$$
\begin{aligned}
I_1 &= \frac{37-\sqrt{633}}{72}Ma^2 \\
I_2 &= \frac{31}{36}Ma^2 \\
I_3 &= \frac{37+\sqrt{633}}{72}Ma^2
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{e}_1 &= [-\frac{-227-9\sqrt{633}}{8(25+\sqrt{633})}, -\frac{-227-9\sqrt{633}}{8(25+\sqrt{633})}, 1]^T \\
\hat{e}_2 &= [-1, 1, 0]^T \\
\hat{e}_3 &= [-\frac{227-9\sqrt{633}}{8(-25+\sqrt{633})}, -\frac{227-9\sqrt{633}}{8(-25+\sqrt{633})}, 1]^T,
\end{aligned} \tag{11}
$$

where $[., ., .]^T$ *stands for a 3D vector in position space.*

(1c) Explicitly (numerically) evaluate the principal axes as unit vectors and compare with those of the the regular cube given in the lecture. Discuss (guess) similarities and differences of the rotation behavior of the regular and cheating dice.

Solution: Evaluating the fractions above and normalizing them to one gives

$$\begin{aligned}\hat{e}_1 &= [0.599, 0.599, 0.530]^T \\ \hat{e}_2 &= [-0.707, 0.707, 0]^T \\ \hat{e}_3 &= [-0.375, -0.375, 0.847]^T, \end{aligned} \tag{12}$$

These are actually not very different from those of Eq. (3.30) of the lecture, which are

$$\begin{aligned}\hat{e}'_1 &= [0.577, 0.577, 0.577]^T \\ \hat{e}'_2 &= [0.707, -0.707, 0]^T \\ \hat{e}'_3 &= [0.333, 0.333, -0.666]^T, \end{aligned} \tag{13}$$

Note, that global signs of vectors do not matter for eigenvectors, so you can multiply entire vectors with -1 while comparing, but not separate components. If we also evaluate the fractions above and compare the principal moment of inertia with lecture notes, it reads as,

$$\begin{aligned}I_1 &= 0.164452Ma^2 \\ I_2 &= 0.863326Ma^2, \\ I_3 &= 0.861111Ma^2, \\ I'_1 &= 0.166Ma^2, \\ I'_2 &= 0.9166Ma^2, \\ I'_3 &= 0.9166Ma^2, \end{aligned} \tag{14}$$

where I'_1, I'_2 and I'_3 are the principal moment of inertia for homogeneous cube for the same mass M. One can see from the comparison that principal axes and moments of inertia are not strongly changed. Since those govern the rotational motion of the dice (week8), it could not be so easy to tell the cheating dice apart from the real one via its behavior.

(2) Stability of rotation: [5pts] Assume you have a rigid body with $\lambda_3 > \lambda_2 > \lambda_1$. Show that rotation around the body axes $\mathbf{e}_1$ and $\mathbf{e}_3$ will be stable, but around $\mathbf{e}_2$ not. *Hint: Stability implies that a small perturbation away from that axis will remain small, instability that it will grow (typically exponentially).*

Solution: As given in the hint, stability of a rotation implies that a small perturbation away from the axis of rotation will remain small. It can be understood from the Euler's equation in lecture notes (Eq. 3.36):

$$\begin{aligned}\dot{\omega}_1 &= \frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_2\omega_3, \\ \dot{\omega}_2 &= \frac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3\omega_1, \\ \dot{\omega}_3 &= \frac{\lambda_1 - \lambda_2}{\lambda_3}\omega_1\omega_2, \end{aligned} \tag{15}$$

where ω_1, ω_2, and ω_3 are the rotation frequencies along principal axes e_1, e_2, and e_3 respectively. Let us start with a simplest case by setting a rigid body to rotate about one of its principle axis, say e_3. In this case, the frequencies ω_1 and ω_2 are zero, which implies that the rigid body rotate with a constant frequency along the principal axis e_3 in the absence of an external torque. This is true for the rotation of a rigid body along any principal axes irrespective of principal moments balance, but we focus on the rotation about the principal axis, e_3 in this discussion.

If we now introduce a small perturbation at $t = 0$ on the rigid body rotating along principal axis e_3 with frequency ω_3, the frequencies ω_1 and ω_2 picks a nonzero values that are, at least initially, small. It can be seen from third equation of **??** *that $\dot{\omega}_3$ remains very small as ω_1 and ω_2 are small. In this case, the first two equation of* **??** *can be written as;*

$$
\begin{aligned}
\ddot{\omega}_1 &\approx \left[\frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_3\right]\dot{\omega}_2 \\
&\approx -\left[\frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1\lambda_2}\omega_3\right]\omega_1 \\
\ddot{\omega}_1 &\approx -\Omega^2\omega_1 \\
\textit{and} \\
\ddot{\omega}_2 &\approx \left[\frac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3\right]\dot{\omega}_1 \\
&\approx -\left[\frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1\lambda_2}\omega_3\right]\omega_2 \\
\ddot{\omega}_2 &\approx -\Omega^2\omega_2, \qquad (16)
\end{aligned}
$$

where $\Omega = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\omega_3/\lambda_1\lambda_2$ is a constant for a constant ω_3 ($\dot{\omega}_3$ is very small). Let us now impose three condition on the value of principal moment λ_3 to check the stability of the motion, which can be later generalized for other two principal moments, λ_1 and λ_2:

*<u>$\lambda_3 > \lambda_2, \lambda_1$</u>: The coefficient Ω^2 is positive nonzero number for this value of λ_3. The solutions of Eq. (**??**) are,*

$$
\begin{aligned}
\omega_1 &= A\cos(\Omega t) + B\sin(\Omega t), \\
\omega_1 &= C\cos(\Omega t) + D\sin(\Omega t), \qquad (17)
\end{aligned}
$$

*where A, B, C, and D are the costants. It is clear from Eq. (**??**) that ω_1 and ω_2 perform a small oscillation around $\omega_1 = \omega_2 = 0$. Therefore, if a rigid body is rotating around the principal axis with largest principal moment, the motion is stable against small perturbations.*

<u>$\lambda_1 < \lambda_3 < \lambda_2$ or $\lambda_2 < \lambda_3 < \lambda_1$</u>: In this case, the coefficient Ω^2 is negative nonzero number,

and the solution of Eq. (??) is a real exponential as,

$$\omega_1 = Ae^{\Omega t} + Be^{-\Omega t},$$
$$\omega_1 = Ce^{\Omega t} + De^{-\Omega t}, \tag{18}$$

where A, B, C, and D are the costants. It can be seen from Fig. **??** *that a small disturbance under this condition can move rapidly away from $\omega_1 = \omega_2 = 0$. The rigid body rotating without any external torque, the rotation about the principal axis with the intermediate principal moment is an unstable motion.*
$\lambda_3 < \lambda_2, \lambda_1$*: Again the coefficient Ω^2 is a positive nonzero number, and ω_1, ω_2 will perform a small oscillation similar to Eq. (??). Therefore, if a rigid body is rotating around the principal axis with smallest principal moment, the motion is stable against small perturbations.*

It can be seen from the above explanation that the stability conditions are true for the rotation of a rigid about any principal axes.

(3) Sceptre on the ground: **[15pts]** Consider a monarch's golden sceptre as shown in figure **??**, which has fallen on the ground and is now circling around the contact point of the handle and the ground with angular velocity Ω as shown.

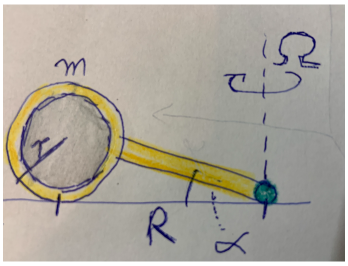

Figure 1: We model the golden sceptre as a hollow sphere of mass m and radius r. The length of the handle (ignore its width) is such that the contact point of the sphere with the ground is a distance R from the contact point of the handle with the ground, as shown.

Solution: To understand the dynamics of the problem, let us start from the scratch. If we look at the symmetry of the golden sceptre, it is not very difficult to notice that one of the principal axis passes along the handle of the sceptre. The principal axis is indicated by $\hat{\mathbf{x}}_3$ in Fig. **??**. *Again from the symmetry of the problem, the other two axes will be in a plane intersecting with the principal axis $\hat{\mathbf{x}}_3$, and perpendicular to it. The axes are labeled as $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_1$, where $\hat{\mathbf{x}}_1$ is into the page and not shown in the Fig.* **??**.

(3a) If the sceptre circles without slipping, find the angular velocity vector that describes its rigid body motion.

Solution: The reason why it is slightly tricky to set up the overall angular velocity, is that there are two different rotations at play here. On the one hand side there must be a component of rotation around the x-axis, since the only fixed point is where the handle

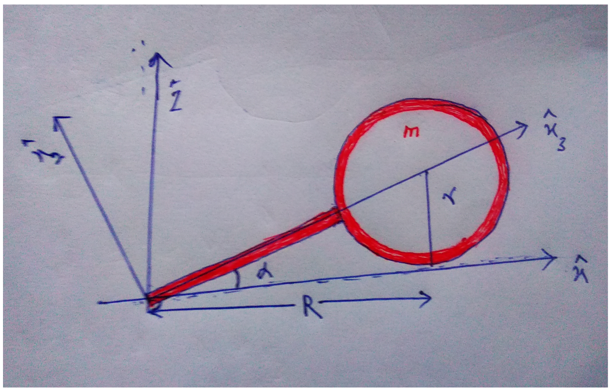

Figure 2: The figure shows a body frame defined by unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, and the principal axes frame defined by unit vectors $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$. The unit vector $\hat{\mathbf{e}}_1$ is into the page and not shown here.

touches the ground and the sceptre circles around that. Since it is rolling without slipping, its main body must simultaneously rotate round the figure symmetry axis as well.

To sort out the combination of these two, we now fix a body frame on golden sceptre whose unit vectors are $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. The two angular velocities discussed above add up, as we had seen in Eq. (2.85) and also used for eq. 3.41 of the lecture notes.

$$\boldsymbol{\omega} = \omega_z \hat{\mathbf{z}} + \omega' \hat{\mathbf{x}}_3, \tag{19}$$

where ω' is the angular velocity of golden sceptre about the main principal axes and ω_z the angular velocity of the principal axes about the space fixed frame (z-axis). From the drawing we can read of that the rotational velocity about the z-axis points downwards (the rotation of the golden sceptre is in the clockwise direction), so we have

$$\boldsymbol{\omega} = -\Omega \hat{\mathbf{z}} + \omega' \hat{\mathbf{x}}_3, \tag{20}$$

The contact points of the sphere of the sceptre form a circle of radius R (see diagram) on the ground. But they also form a circle of radius $r\cos\alpha$ on the sphere, where α is the angle between the handle and the ground as indicated in Fig. **??**. *Since the golden sceptre is circling without slipping, $\Omega R = \omega'(r\cos\alpha)$, and thus*

$$\begin{aligned}
\boldsymbol{\omega} &= -\Omega \hat{\mathbf{z}} + \frac{\Omega R}{r\cos\alpha}\hat{\mathbf{x}}_3 \\
&= -\Omega \hat{\mathbf{z}} + \frac{\Omega R}{r\cos\alpha}(\cos\alpha \hat{\mathbf{x}} + \sin\alpha \hat{\mathbf{z}}) \\
&= -\Omega \hat{\mathbf{z}} + (\frac{\Omega R}{r}\hat{\mathbf{x}} + \Omega \hat{\mathbf{z}}), \\
\boldsymbol{\omega} &= \frac{\Omega R}{r}\hat{\mathbf{x}},
\end{aligned} \tag{21}$$

where $\tan\alpha = r/R$ is used in the calculation. Note that the form above is just at a snapshot in time where the sceptre axis is within the xz plane. (but we can always just adjust our *xyz coordinate system such that this is the case.*

(3b) Find the normal force between the ground and the sceptre.

Solution: We shall now find the time evolution of the angular momentum of the sceptre. That turns out not to be constant, hence the ground must be exerting a force on the sceptre, which leads to a torque.

The components of the angular velocity along the two principal axis $\hat{\mathbf{x}}_2$ *and* $\hat{\mathbf{x}}_3$ *are*

$$\begin{aligned} \omega_2 &= -\frac{R}{r}\Omega \sin\alpha, \\ \omega_3 &= \frac{R}{r}\Omega \cos\alpha. \end{aligned} \tag{22}$$

The principal moments are

$$\begin{aligned} I_3 &= \frac{2}{3}mr^2, \\ I_2 &= \frac{2}{3}mr^2 + m(r^2 + R^2). \end{aligned} \tag{23}$$

where I_3 *is calculated by using the parallel axis theorem. The angular momentum vector of the golden sceptre is* $\mathbf{L} = I_2\omega_2\hat{\mathbf{x}}_2 + I_3\omega_3\hat{\mathbf{x}}_3$*, so its horizontal component has magnitude of* $L_h = I_3\omega_3 \cos\alpha - I_2\omega_2 \sin\alpha$*. Therefore, the magnitude of* $d\mathbf{L}/dt$ *is*

$$\begin{aligned} \left|\frac{d\mathbf{L}}{dt}\right| &= L_h\Omega = (I_3\omega_3\cos\alpha - I_2\omega_2\sin\alpha)\Omega \\ &= \left[\left(\frac{2}{3}mr^2\right)\left(\frac{R}{r}\Omega\cos\alpha\right)\cos\alpha \right. \\ &\left. - \left(\frac{2}{3}mr^2 + m(r^2+R^2)\right)\left(-\frac{R}{r}\Omega\sin\alpha\right)\sin\alpha\right]\Omega \\ &= \left[(\frac{2}{3}mr^2\right)\left\{\frac{R}{r}\Omega\right\} + m(r^2+R^2)\left(\frac{R}{r}\Omega\sin^2\alpha\right)\right]\Omega \\ &= \frac{m\Omega^2 R}{r}\left[\frac{2}{3}r^2 + r^2\right] \\ &= \frac{5}{3}m\Omega^2 rR, \end{aligned} \tag{24}$$

where we have used $\sin\alpha = r/\sqrt{r^2+R^2}$ *in third last line of Eq. (**??**). The direction of* $d\mathbf{L}/dt$ *is out of the page. The torque (relative to the pivot) is due to the gravitational force acting at the CM, along with the normal force* N *acting at the contact point. Therefore,* τ *points out of the page with magnitude* $|\tau| = (N - mg)R$*. Equating* $|\tau|$ *with* $d\mathbf{L}/dt$ *in*

Eq. (??) gives,

$$(N - mg)R = \frac{5}{3} m\Omega^2 rR,$$

$$N = mg + \frac{5}{3} m\Omega^2 r. \tag{25}$$

This has the interesting property of being independent of R, and hence also α.

(4) Inertia tensor: [10pts] You can see from many examples in books that finding moments of inertia analytically for complicated objects can be challenging. One way that always works, is numerical integration. A very simple method for numerical integration, is simply approximating the integral by a bunch of rectangles and writing $\int_a^b f(x)dx \approx \sum_k f(x_k)\Delta x$, where x_k are a set of equally spaced points between a and b with spacing Δx.

The template `Assignment5_program_draft_v1.m` allows you to generate quite complicated rigid bodies assembled out of spheres, cuboids or cylinders, but starts with a predefined "snow-figure". It should then numerically find the moment of inertia tensor (matrix) and diagonalize that matrix (again numerically) to show you the principal axes. The script is long due to lots of visualisations. You may ignore most of it unless interested.

(4a) The first element of the moment of inertia tensor I_{xx} is already implemented. Implement the other ones and run the script (you can search for `POSITION 3` in the script to find the place).

(4b) If you change the number of elements in the rigid body from 5 to 3 (`POSITION 1`), it will skip arms and nose. How does this affect the principal axes? Why?

Solution: The rigid body with its principal axes for element 5 and element 3 is shown in Fig. **??** *and Fig.* **??** *respectively. To understand the affect on the principal axes from element 5 to 3, let us first start from the rigid body for element 3.*

It can be seen from Fig. **??** *that the rigid body is perfectly symmetric about an axis passing through it along the z-direction. Therefore, this axis is one of the principal axis for the rigid body as shown in Fig.* **??***. Other two axes should be in a perpendicular plane intersecting this axis, which happened to be in $X - Y$ plane at $z = 0$ for this case. The principal axes reads as,*

$$\begin{aligned} \hat{e}_1 &= [0.8470, -0.5316, 0]^T, \\ \hat{e}_2 &= [0.5316, 0.8470, 0]^T, \\ \hat{e}_3 &= [0, 0, 1]^T. \end{aligned} \tag{26}$$

The rigid body in Fig. **??** *gets arms and nose, which shift the symmetry of the rigid compared to Fig.* **??***. It can be seen from the Fig.* **??** *that due to arms and nose, the rigid body is almost perfectly symmetric along axes passing through it along z and x-directions. Therefore, these axes are the principal axes as shown in the Fig.* **??***. Now the third* *principal axis is trivially along the Y direction. It can also be seen from the calculation:*

$$\hat{e'}_1 = [-1, 0, 0.0078]^T,$$
$$\hat{e'}_2 = [0, 1, 0]^T,$$
$$\hat{e'}_3 = [-0.0078, 0, -1]^T. \tag{27}$$

So a change in the symmetry of a rigid body will result in shift of principal axes.

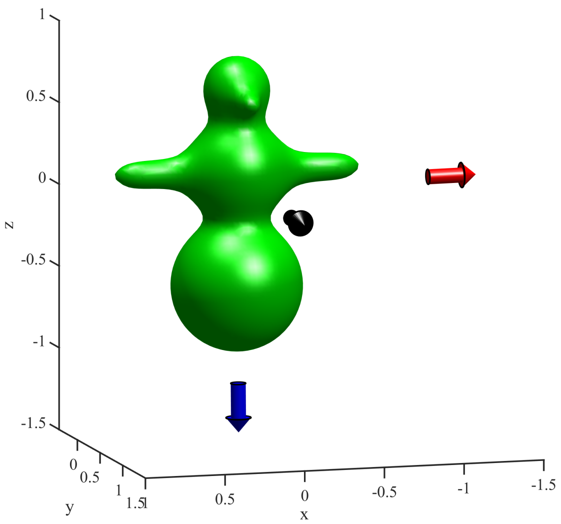

Figure 3: The figure shows a rigid body for element 5 in the matlab script. The principal axes are indicated by red, black, and blue arrows.

(4c) Now play a bit with the rigid body/figure, by roughly understanding how it is defined (`POSITION 2`) and moving objects a bit by changing their centre (keep all coordinates within $-1 \cdots 1$). If you reduce the symmetry of the figure, what happens to the principal axes? How does the magnitude of the moments of inertia reflect the shape of the object?

Solution: The rigid body in Fig. **??** *is drastically disturbed from the symmetry present in Fig.* **??** *and Fig.* **??***. As we discussed in 4 (b), the principal axes of a rigid body are the virtue of its geometrical distribution. Therefore, the frame associated with the principal axes ($\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$) is tilted by an angle with respect to body fixed frame ($\hat{x}, \hat{y}$ and $\hat{z}$) compared to Fig.* **??** *and Fig.* **??** *because its geometrical distribution is not symmetric along the axes of body frame. The principal axes for this case reads as,*

$$\hat{e'}_1 = [-1, 0.0022, 0.0066]^T,$$
$$\hat{e'}_2 = [-0.0003, 0.9343, -0.3565]^T,$$
$$\hat{e'}_3 = [-0.0070, 0.3565, 0.9343]^T. \tag{28}$$

Since moments of inertia also depend on the distribution of mass around the axis of rotation, the change in the symmetry of a rigid body will alter its magnitude. The

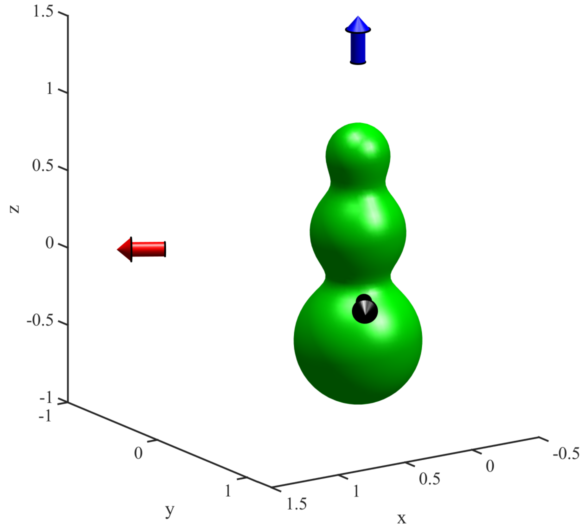

Figure 4: The figure shows a rigid body for element 3 in the matlab script. The principal axes are indicated by red, black, and blue arrows.

magnitude of principal moments for three cases can be written as:

$$I_{??} = \begin{bmatrix} 0.2709 & 0 & 0 \\ 0 & 0.2115 & 0 \\ 0 & 0 & 0.1175 \end{bmatrix},$$

$$I_{??} = \begin{bmatrix} 0.1792 & 0 & 0 \\ 0 & 0.1680 & 0 \\ 0 & 0 & 0.0508 \end{bmatrix},$$

$$I_{??} = \begin{bmatrix} 0.1637 & 0 & 0 \\ 0 & 0.1637 & 0 \\ 0 & 0 & 0.0372 \end{bmatrix},$$

where $I_{??}, I_{??}$, and $I_{??}$ are the principal moments for Fig. **??**, *Fig.* **??**, *and Fig.* **??** *respectively.*

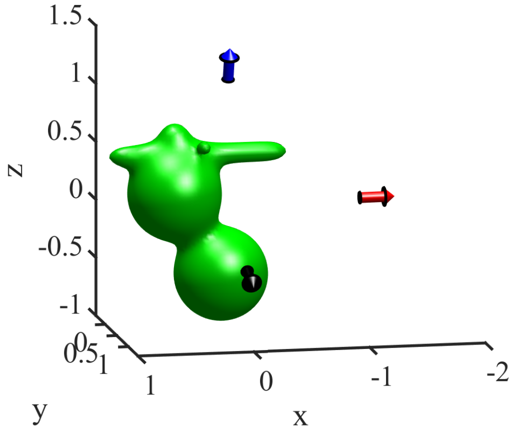

Figure 5: The figure shows a rigid body for element 5 in the matlab script. The symmetry of the rigid body has been disturbed by shifting the center and changing the length of the shape in different elements. The principal axes are indicated by red, black, and blue arrows.