PHY 305, I-Semester 2020/21, Assignment 4

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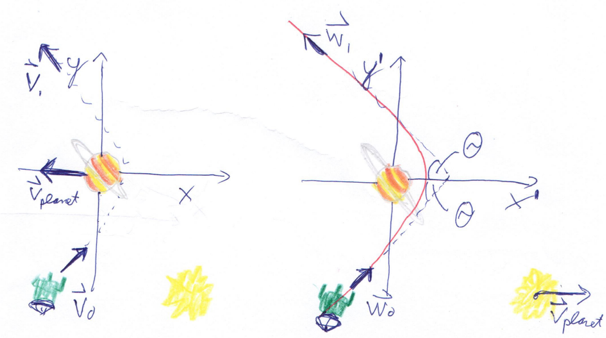
Due-date: 03.10.2020

(1) Gravity assist: [10pts]
When sending space-probes out of the solar system (Pioneer, Voyager), humanity has used a trick called "gravity-assist" to help the probes reach solar escape velocity. As sketched in the figure below, for this the space-probe does a close fly-by near a heavy outer planet such as Saturn, changing its travelling direction while doing so.

Let the velocities in a sun fixed reference frame be $\mathbf{v}_0$ (probe in) $\mathbf{v}_1$ probe out and $\mathbf{V}_{\text{planet}}$ for the planet. Consider the situation where in a frame fixed on the planet the probe comes in on a hyperbolic Kepler orbit with eccentricity ϵ, such that the asymptotic trajectory far from the planet makes an angle θ with the x-axis, as shown. We denote the velocities in this frame by $\mathbf{w}_{0,1}$.



left: (left) Space probe fly-by past Saturn in a solar system fixed frame. (right) The same in a planet fixed frame. The sun is drawn for orientation in the bottom right corner, but not important in this question.

(1a) Show that $\theta = \pi - \arccos(-1/\epsilon)$.

Solution: From Kepler's law for an elliptical orbit of a planet around a star is given by,

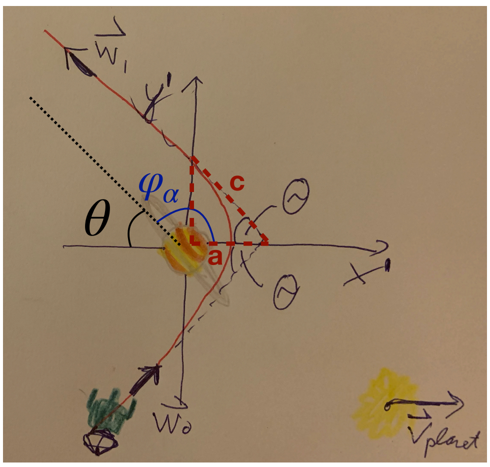


Figure 1: Sketch

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\varphi)}, \tag{1}$$

where r is the distance of the planet from the star, ϵ is the eccentricity, a is the semimajor axis of the orbit and φ is the angle made by the distance vector r with x axis. When $r \to \infty$ we write,

$$1 + \epsilon\cos(\varphi_\infty) = 0, \tag{2}$$

$$\Rightarrow \cos(\varphi_\infty) = -\frac{1}{\epsilon} \tag{3}$$

where $\varphi = \varphi_\infty$ when $r \to \infty$. From Fig. (1) we can see that $\pi = \theta + \varphi_\infty$. Hence we get,

$$\theta = \pi - \arccos(-1/\epsilon) \tag{4}$$

(1b) Using this, show that in the solar system frame $v_1^2 - v_0^2 = 4w_0V/\epsilon > 0$, where e.g. $v_0 = |\mathbf{v}_0|$ etc. This means the space probe is accelerated by its fly-by past the planet. Where does the required energy come from?

Solution: In the solar frame we can write,

$$\mathbf{v}_{\mathit{0}} = \mathbf{w}_{\mathit{0}} + \mathbf{V}_{\mathit{planet}}, \tag{5}$$

$$\mathbf{v}_{\mathit{1}} = \mathbf{w}_{\mathit{1}} + \mathbf{V}_{\mathit{planet}}. \tag{6}$$

Therefore,

$$\begin{aligned}|\mathbf{v}_{\mathit{1}}|^2 - |\mathbf{v}_{\mathit{0}}|^2 =& \Big(\mathbf{w}_{\mathit{1}}^2 + \mathbf{V}_{\mathit{planet}}^2 - 2\,\mathbf{w}_{\mathit{1}}\cdot\mathbf{V}_{\mathit{planet}}\Big) \\ &- \Big(\mathbf{w}_{\mathit{0}}^2 + \mathbf{V}_{\mathit{planet}}^2 - 2\,\mathbf{w}_{\mathit{0}}\cdot\mathbf{V}_{\mathit{planet}}\Big).\end{aligned} \tag{7}$$

From the energy conservation we know that,

$$|\mathbf{w}_{\mathit{0}}| = |\mathbf{w}_{\mathit{1}}|. \tag{8}$$

Hence we get,

$$|\mathbf{v}_{\mathit{1}}|^2 - |\mathbf{v}_{\mathit{0}}|^2 = 2\Big(\mathbf{w}_{\mathit{1}} - \mathbf{w}_{\mathit{0}}\Big)\cdot\mathbf{V}_{\mathit{planet}} \tag{9}$$

From figure we see that θ is angle between $\mathbf{w}_{\mathit{1}}, \mathbf{V}_{\mathit{planet}}$ and also between $-\mathbf{w}_{\mathit{0}}, \mathbf{V}_{\mathit{planet}}$. Hence,

$$\mathbf{w}_{\mathit{0}}\cdot\mathbf{V}_{\mathit{planet}} = w_0\,V_{\mathit{planet}}\,\cos(\theta), \tag{10}$$

$$\mathbf{w}_{\mathit{1}}\cdot\mathbf{V}_{\mathit{planet}} = -w_1\,V_{\mathit{planet}}\,\cos(\theta). \tag{11}$$

Hence we can rewrite Eq.(9) as,

$$2\Big(\mathbf{w}_{\mathit{1}} - \mathbf{w}_{\mathit{0}}\Big)\cdot\mathbf{V}_{\mathit{planet}} = 4\,w_0 V_{\mathit{planet}}\,\cos(\theta) \tag{12}$$

Since $\cos(\theta) = \cos(\pi - \arccos(-1/\epsilon)) = 1/(\epsilon)$ *we have,*

$$|\mathbf{v}_1|^2 - |\mathbf{v}_0|^2 = \frac{4\, w_0 V_{planet}}{\epsilon} > 0 \tag{13}$$

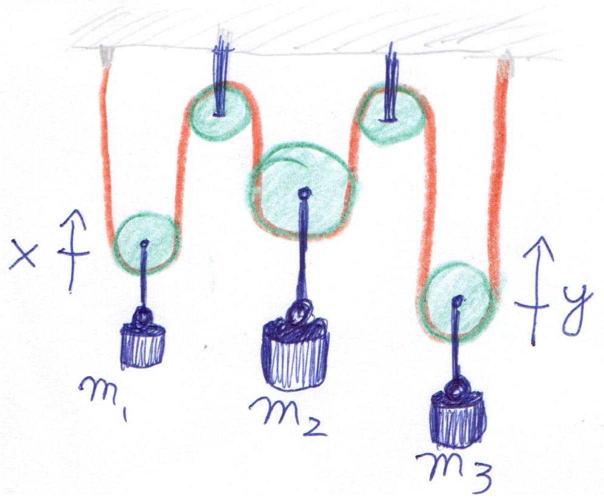
(1c) What happens when the direction of planet motion is opposite to that shown in the figure (with all other data identical)?

Solution: If the direction of planet motion is opposite it changes the sign of the scalar products in Eq. (10) and (11). Hence,

$$|\mathbf{v}_1|^2 - |\mathbf{v}_0|^2 = -\frac{4\, w_0 V_{planet}}{\epsilon} < 0. \tag{14}$$

This means the probe will be decelerated.

(2) Ropes and Pulleys: [10pts] Consider the arrangement of a single rope, threaded through five pulleys shown below. It is rigidly attached to the ceiling but each pulley can freely roll within the rope. Three weights are hung on the lower pulleys as indicated, with $m_1 = 5m_0$, $m_2 = 3m_0$, $m_3 = 7m_0$, where m_0 is some reference mass.



left: Sketch of rope, weight and pulleys as described in the text

(2a) Setup the Lagrangian in the generalised coordinates given by x and y, which are the upwards motion of weights m_1 and m_3 relative to their initial position.

Solution: If the left mass m_1 *goes up by* x *and the right mass* m_2 *by* y *then the middle mass* m_3 *will go down by* $x{+}y$*. Therefore the Lagrangian of the system,*

$$\begin{aligned} L =& \frac{1}{2}(5m_0)\dot{x}^2 + \frac{1}{2}(3m_0)(-\dot{x}-\dot{y})^2 + \frac{1}{2}(7m_0)\dot{y}^2, \\ &- \Big(5m_0 g x + 3m_0 g(-x-y) + 7m_0 g y\Big), \end{aligned} \tag{15}$$

$$L = 4m_0\dot{x}^2 + 3m_0\dot{x}\dot{y} + 5m_0\dot{y}^2 - 2m_0 g(x+2y) \tag{16}$$

(2b) Find at least one continuous symmetry of that Lagrangian and then use Noether's theorem to identify the associated conserved quantity.

Solution: The Lagrangian in Eq. (16) will be invariant under the transformation $x \to x + 2\epsilon$ *and* $y \to y - \epsilon$. *Hence if we use the Noether's theorem with* $Q_x = 2$ *and* $Q_y = -1$ *the conserved momentum will be,*

$$\frac{\partial L}{\partial \dot{x}} Q_x + \frac{\partial L}{\partial \dot{y}} Q_y = m_0(8\dot{x} + 3\dot{y})(2) + m_0(10\dot{x} + 3\dot{y})(-1), \tag{17}$$

$$\Rightarrow \; 3m_0(2\dot{x} + \dot{y}) = const. \tag{18}$$

(2c) Derive Lagrange equations and from those verify explicitly that the quantity is indeed conserved.

Solution: The EL equation will be,

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -2m_0 g - 8m_0\ddot{x} - 3m_0\ddot{y} = 0, \tag{19}$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = -4m_0 g - 10m_0\ddot{x} - 3m_0\ddot{y} = 0 \tag{20}$$

$$8m_0\ddot{x} + 3m_0\ddot{y} = -2m_0 g, \tag{21}$$

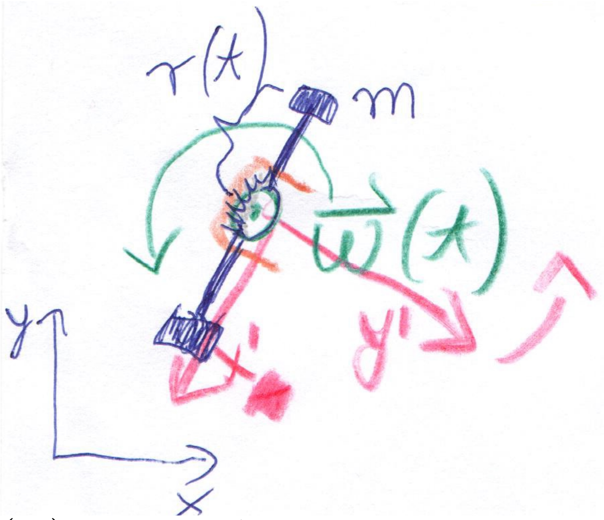
$$10m_0\ddot{x} + 3m_0\ddot{y} = -4m_0 g \tag{22}$$

Subtracting $2\times$*(21) from (22) gives,*

$$6m_0\ddot{x} + 3m_0\ddot{y} = 0 \quad \Rightarrow \frac{d}{dt}\Big(6m_0\dot{x} + 3m_0\dot{y}\Big) = 0, \tag{23}$$

$$\Rightarrow \; 3m_0(2\dot{x} + \dot{y}) = const. \tag{24}$$

(3) Swivel-chair: [10pts] Watch the following videos related to angular momentum conservation (video1 , video2) and refer to the diagram below.



left: Sketch of professor (blue) on swivel chair (brown), viewed from the top. We also show the lab-frame (blue) and co-rotating frame (red). We assume the professor to be massless but carrying heavy weights of mass m in each hand. The distance from hands to rotation axis is $r(t)$.

(3a) Using angular momentum conservation (why can you use it?), find the rotation velocity $\Omega(t)$ (green) as a function of the initial rotational velocity $\Omega_0 = \Omega(t=0) > 0$ and the initial distance of the weights from the body $r_0 = r(t=0) > 0$ and the unspecified function $r(t)$.

Solution: Since there is no external torque acting on the system we can use angular momentum conservation. Using the definition $\mathbf{L} = \mathbf{r} \times \mathbf{p}$*, the fact that* $|\mathbf{L}| = rp$ *since* $\mathbf{r}$ *and* $\mathbf{p}$ *are orthogonal and* $p = mv = mr\Omega$*, we reach*

$$const = L = 2mr_0^2\,\Omega_0 = 2mr(t)^2\,\Omega(t), \tag{25}$$

$$\Rightarrow \Omega(t) = \left(\frac{r_0^2}{r(t)}\right)^2 \Omega_0. \tag{26}$$

Thus, as the person on the chair pulls the hands with the weights in ($r(t)$ reduces from its initial value r_0), the rotational velocity increases quadratically with $r(t)$.

Note that pulling the hands in does not give rise to a torque $\mathbf{N}$ *around the origin, since* $\mathbf{N} = \mathbf{r} \times \mathbf{F}$ *and* $\mathbf{F}$ *and* $\mathbf{r}$ *are parallel, in which case their cross product vanishes.*

(3b) In our derivation of the fictitious forces in a rotating coordinate system in section 2.9.2 (Eq. 2.87), we had assumed that the rotation axis does not depend on time. Generalize the derivation for the case where $d\boldsymbol{\Omega}/dt \neq 0$. Give an equation for the additional fictitious force that you find.

Solution: If we re-trace the derivation of 2.87 for the case $d\boldsymbol{\Omega}/dt \neq 0$ *we see that (Eq. 2.85) still holds unchanged:*

$$\left(\frac{\partial \boldsymbol{Q}}{dt}\right)_{S_0} = \left(\frac{\partial \boldsymbol{Q}''}{dt}\right)_{S''} + \boldsymbol{\Omega}(t) \times \boldsymbol{Q}''. \tag{27}$$

However when we apply this to the conversion of Newton's equation, we get additional terms, since for the second application we get time-derivatives acting on $\boldsymbol{\Omega}(t)$ *that we had* *previously set to zero:*

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S''}\left[\left(\frac{d\boldsymbol{r''}}{dt}\right)_{S''} + \boldsymbol{\Omega(t)} \times \boldsymbol{r''}\right] + \boldsymbol{\Omega(t)} \times \left[\left(\frac{d\boldsymbol{r''}}{dt}\right)_{S''} + \boldsymbol{\Omega(t)} \times \boldsymbol{r''}\right]. \tag{28}$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S_0} = \ddot{\mathbf{r}}'' + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}}'' + \dot{\boldsymbol{\Omega}} \times \mathbf{r}'' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}''). \tag{29}$$

The additional fictitious force arising from the third term on the right is known as azimuthal force and given by

$$\boldsymbol{F}_{azimuthal} = m\,\dot{\boldsymbol{\Omega}} \times \mathbf{r}''. \tag{30}$$

(3c) Now rederive the result of (3a), using fictitious forces in the frame fixed on the person on the chair (see figure).

Solution: The fictitious forces will be,

$$\boldsymbol{F} = \underbrace{2m\,\boldsymbol{\Omega}(t) \times \dot{\boldsymbol{r}}''}_{=\mathbf{F}_{cor}} + \underbrace{m\,\dot{\boldsymbol{\Omega}}(t) \times \boldsymbol{r}''}_{=\boldsymbol{F}_{azimuthal}} + \underbrace{m\,\boldsymbol{\Omega}(t) \times (\boldsymbol{\Omega}(t) \times \boldsymbol{r}'')}_{=\mathbf{F}_{cf}}. \tag{31}$$

The centrifugal force $\mathbf{F}_{cf}$ will be pointing radially outwards and hence has to be overcome by the forces exerted by the arms to pull the hands/weights inwards. The Coriolis force $\mathbf{F}_{cor}$ and azimuthal force from part (b) both point in the azimuthal direction (in the xy plane and perpendicular to $\mathbf{r}''$. However they must both cancel, since otherwise there would be a <u>sideways force</u>, which would cause sideways motion and hence rotation. However that is a contradiction to us saying that we work in the frame rotating with the weights/hands. If we build in that Ω is increasing (and hence $\dot{\boldsymbol{\Omega}}(t)$ parallel to $\boldsymbol{\Omega}(t)$) and the hands moving inwards (so $\dot{\mathbf{r}}'' \sim -\mathbf{r}$), you can convince yourself with the right hand rule and a diagram, that $\mathbf{F}_{cor}$ and $\boldsymbol{F}_{azimuthal}$ are pointing in opposite directions. Since all vectors in the cross products are orthogonal, we can finally write

$$2m\Omega(t)\underbrace{(-\dot{r}'')}_{=|\dot{r}''|} = m\dot{\Omega}(t)r'' \tag{32}$$

After multiplying both sides by r'' this again states that

$$\frac{d}{dt}(m\Omega(t)r''(t)^2) = 0, \tag{33}$$

i.e. angular momentum is conserved, and we thus reach the same relation as in (3a).

(4) Solar system simulator: [10pts] Formulate Newton's equations for all the major bodies in the solar system. The template `Assignment4_program_draft_v1.m` solves these for the subset (sun, earth, mars) in two dimensions (x,y).

(4a) Find the initial conditions that place earth and mars on the correct orbit (see internet) and verify them, using `assignment4_solarsystem_slideshow_v1.m`, for example by checking for the correct orbital period.

Solution:

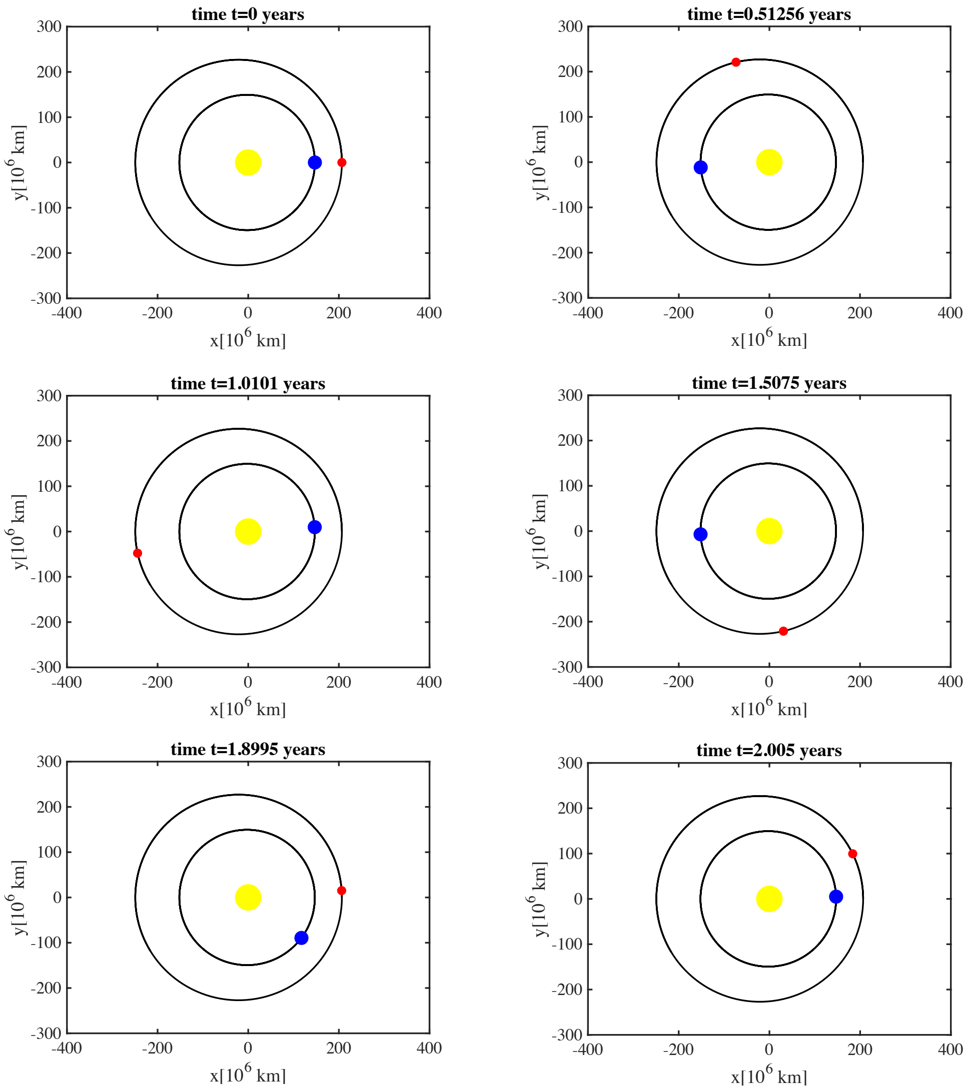


Figure 2: Orbiting Earth and Mars around Sun.

(4b) With the script `assignment4_radialevolution_draft_v1.m`, you can check the time evolution of the radial distance of mars from the sun. Edit the script so that panel (b) plots the effective radial potential $V_{\text{eff}}(r)$ and the energy of mars and explain the evolution of the radial distance with it. Then make Mars's orbit more eccentric, by reducing its initial velocity with an arbitrary scale factor. What happens?

Solution: The effective potential for mars with mass m and sun with mass M is given by,

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{GmM}{r}, \tag{34}$$

where the second term in V_{eff} is the gravitational potential and first term corresponds to the centrifugal potential. Now with this effective potential the radial distance of mars from the sun oscillates with time i.e the mars revolves around the sun changing it's radial distance periodically as it moves along the elliptical orbit. Now if we reduce the initial velocity the frequency of oscillation increases.

(4c) [Bonus] Change the simulator to a triple-star system by changing earth and mars masses to equal that of the sun. What happens? Can you engineer a "stable" triple-star system?

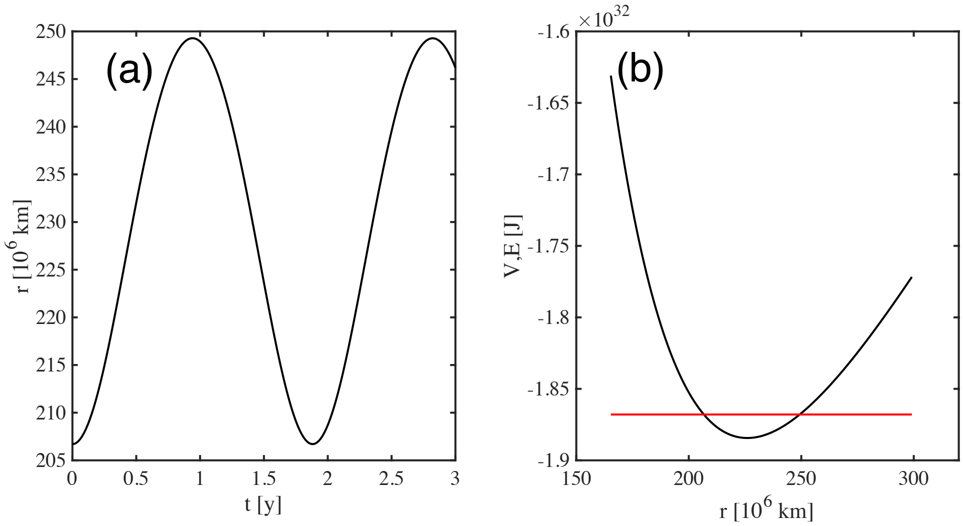


Figure 3: (a) Time evolution of radial distance of Mars. (b) Effective potential.

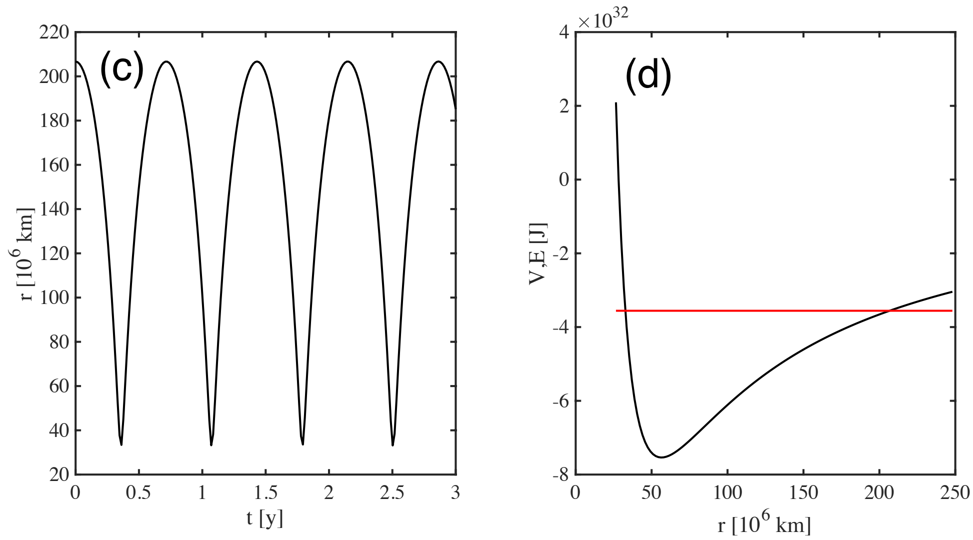


Figure 4: (c) Time evolution of radial distance of Mars after reducing the velocity to half. (d) Effective potential after reducing the velocity.

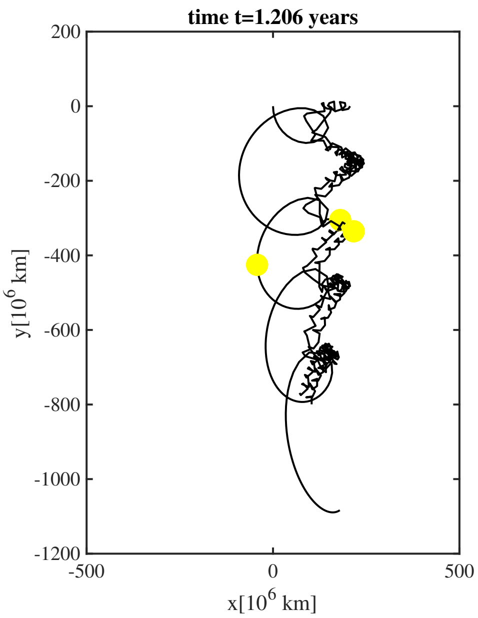


Figure 5: Triple sun system.