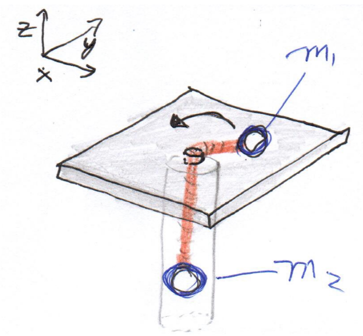
PHY 305, I-Semester 2020/21, Assignment 3 solution

(1) Constrained system: [10pts]

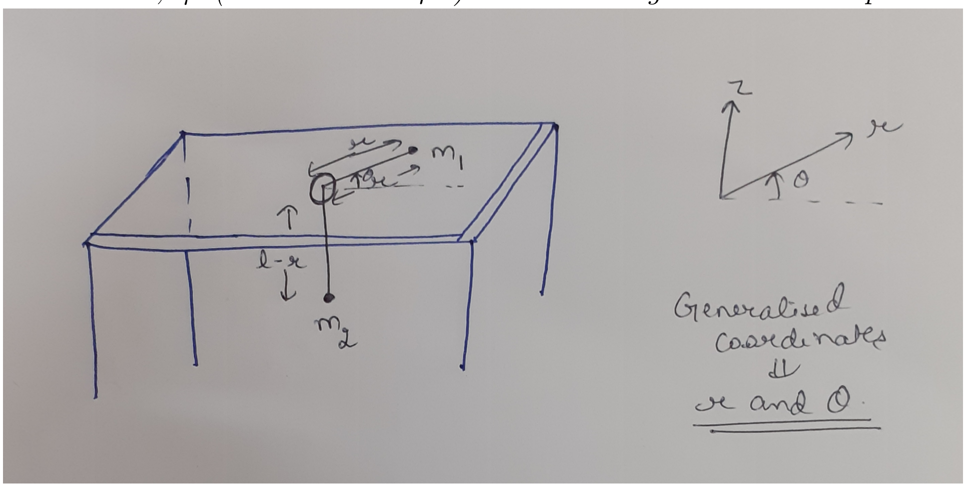


left: A ball of mass m_1 can move without friction on a surface (grey) which forms the xy plane. It is connected through a rope which is threaded through a hole in the surface with a mass m_2 hanging below the surface. The latter is confined in a plexiglass tube, so that it can only move along the z direction. Consider only cases without slack in the rope and it being straight on the table, as in the figure

(1a) Explicitly write the constraint equations for this problem, then discuss how many degrees of freedom the problem has and which generalised coordinates you would propose.

Solution: Without constraints 2 balls in 3 dimension would be 6 degrees of freedom. Lets call these x_1, y_1, z_1 for mass 1 and x_1, y_2, z_3 for mass 2. The first constraint is that mass m_1 must move on the surface, hence $z_1 = 0$ (we place $z = 0$ on the surface). The second constraint is that mass m_2 must move in the tube, hence $x_2 = 0$ and $y_2 = 0$ (we place the origin of the xy plane at the tube. All these constraint equations already take the form (2.21) of the lecture). The final constraint is that the length ℓ of the string is constant. Thus we can write a final constraint equation in the remaining coordinates as $\sqrt{x_1^2 + y_1^2} + |z_2| = \ell$. We started with 6 DGF and then found 4 constraint equations which leaves us with two degrees of freedom.

We now switching to polar coordinates for the motion of mass 1 and hence have coordinates r, φ (see section 1.4.6). See drawing below but swap $\theta \leftrightarrow \varphi$ in the drawing.



We can rewrite the last constraint equation as $z = -(\ell - r)$ and thus express it in terms of r. So r, φ are a useful set of generalized coordinates. From $z = -(\ell - r)$ we can also directly deduce $\dot{z} = \dot{r}$.

(1b) Write the Lagrangian for this problem.

Solution: Using $x = r\cos\varphi$, $y = r\sin\varphi$, *the kinetic energy of mass* m_1 *is*

$$T_1 = \frac{1}{2}m_1(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\varphi}^2)$$

and that of mass m_2 *is*

$$T_2 = \frac{1}{2}m_2\dot{z}^2 = \frac{1}{2}m_2\dot{r}^2$$

.

Thus, the total kinetic energy of the system is

$$T = T_1 + T_2$$

$$\Rightarrow T = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{1}{2}m_2\dot{r}^2 \tag{1}$$

Taking the table as the reference, the Potential energy of mass m_1 *is*

$$V_1 = 0$$

and that of mass m_2 *is*

$$V_2 = m_2\, g\, z$$

$$\Rightarrow \qquad V_2 = -m_2\, g\,(l - r)$$

We arrive at a total potential energy of the system

$$V = V_1 + V_2$$

$$\Rightarrow V = -m_2\, g\,(l - r) \tag{2}$$

Finally, the Lagrangian L *is given by*

$$L = T - V$$

Using (2) and (3) ,

$$\Rightarrow L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{1}{2}m_2\dot{r}^2 + m_2\, g\,(l - r). \tag{3}$$

(1c) Find the Lagrange equations, identify conditions where mass m_2 can be at rest, and find the frequency of small oscillations around these conditions.

Solution: Writing down the Lagrange equation for r *gives*

$$(m_1 + m_2)\ddot{r} = m_1\, r\,\dot{\varphi}^2 - m_2\, g. \tag{4}$$

This equation says that the acceleration of the two masses along the direction of the string is determined by a balance of the gravitational force acting on mass 2 and the centrifugal force of m_1*. The Lagrange equation for* φ *is*

$$\frac{d}{dt}(m_1\, r^2\,\dot{\varphi}) = 0.$$

This equation says that the angular momentum $L_z = m_1 r^2 \dot{\varphi}$ is conserved. We can use this knowledge to eliminate $\dot{\varphi}$ from Eq. 4 to reach

$$(m_1 + m_2)\ddot{r} = \frac{L_z^2}{m_1 r} - m_2 g. \tag{5}$$

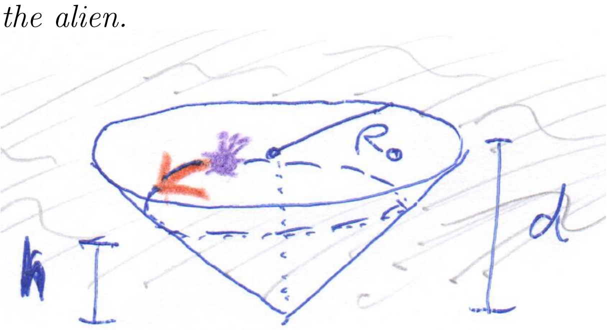
Conditions where m_2 remains at rest must mean that $\dot{z} = 0$ and also $\ddot{z} = 0$. Due to the constraint $z = -(\ell - r)$ this implies that also $\dot{r} = 0$ and also $\ddot{r} = 0$, hence from Eq. 5 the sought conditions are $r = \frac{L_z^2}{m_1 m_2 g} \equiv r_0$. To find the frequency of small oscillations, we Taylor expand the rhs. of (4) near r_0 to find

$$(m_1 + m_2)\ddot{r} = -\left(\frac{g^2 m_1 m_2^2}{L_z^2}\right)(r - r_0). \tag{6}$$

By comparison with the differential equation for the simple harmonic oscillator $\ddot{x} = -\omega^2 x$, we infer for the (angular) frequency for oscillations around the equilibrium positions:

$$\omega = \sqrt{\frac{g^2 m_1 m_2^2}{L_z^2 (m_1 + m_2)}}. \tag{7}$$

(2) Alien fun: [10pts] *Europa is a moon of Jupiter with a surface made of ice. An asteroid impact has made a perfectly shaped impact crater as shown in the figure below. An alien life form living on Europa wants to entertain itself by using the crater as a slide, frictionlessly sliding as shown in the figure such that it remains a constant height h over the bottom of the crater, despite Europa's gravity with surface acceleration g_E acting on the alien.*



left: *Sketch of impact crater (blue). It has a radius R_0 at the top and a total depth of d, in between the radius shrinks linearly. The alien is violet, sliding in a direction ini*

(2a) *Setup the Lagrangian in useful generalised coordinates, and find the Lagrange equations.*

Solution: Let r and θ be the generalised coordinates, where r be the distance from the axis and φ be the angle around the cone.
So, if α is the semi-vertical angle of the cone then,

$$\tan\alpha = R_0/d \tag{8}$$

where R_0 is the radius of the cone at the top and d is the depth of the cone.
If r_0 is the radius of the cone corresponding to height h, then we have

$$\tan\alpha = r_0/h \tag{9}$$

So, from (5) and (6);

$$R_0/d = r_0/h$$

$$\Rightarrow r_0 = R_0 h/d \tag{10}$$

Now, if r is the distance of the particle from the axis, then the distance of the particle up along the cone is $r/\sin\alpha$ *and hence the component of the velocity up along the cone is* $\dot{r}/sin\alpha$ *and the component of the velocity around the cone is* $r\dot{\varphi}$.
So, the square of the speed is

$$v^2 = \dot{r}^2/\sin^2\alpha + r^2\dot{\varphi}^2$$

. The kinetic energy is therefore

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\varphi}^2\right) \tag{11}$$

Since r and φ *are the generalised coordinates, we have to express our Lagrangian in terms of r and* φ.
So the height from the bottom = $r/\tan\alpha$ *corresponding to distance r from the axis of the cone and hence Potential energy is*

$$V = mgh = \frac{mgr}{\tan\alpha} \tag{12}$$

From equation (8) and (9), the Lagrangian L is given by

$$L = T - V \Rightarrow L = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\varphi}^2\right) + \frac{mgr}{\tan\alpha} \tag{13}$$

Now, using Lagrange equation; the equation of motion are as follows:
For r:

$$\ddot{r} = r\dot{\varphi}^2\sin^2\alpha - g\cos\alpha\sin\alpha \tag{14}$$

For φ*:*

$$\frac{d}{dt}(mr^2\dot{\varphi}) = 0$$

$$\Rightarrow mr^2\dot{\varphi} = L \;\; (constant) \tag{15}$$

L stands for angular momentum.

(2b) *What is the frequency* ω *with which the Alien will complete its circles?*

Solution: We need to find the frequency ω *with which the Alien complete its circle of radius* r_0 *at height h from the bottom of the cone.*
Since the radius of the circle = r_0 *is fixed, therefore*

$$\dot{r} = \ddot{r} = 0$$

Hence, equation (11) yields

$$r\dot{\varphi}^2\sin^2\alpha - g\cos\alpha\sin\alpha = 0$$

$$\Rightarrow r\,\dot{\varphi}^2 \sin^2\alpha = g\cos\alpha\sin\alpha \Rightarrow \omega = \dot{\varphi} = \sqrt{\frac{g}{r_0\tan\alpha}}$$

Using equation (7),

$$\omega = \dot{\varphi} = \sqrt{\frac{g\,d}{R_0\,h\tan\alpha}} \tag{16}$$

Hence, the frequency with which the Alien moves in its circle is given by equation(13).

(2c) *If it slightly misses the height h, what will be the frequency of oscillations around that height h?*

Solution: We need to find the frequency of oscillations around height h. This is also the frequency around radius r_0.Let Ω be this frequency.
Using the value of L from equation (12) into equation (11), we can get;

$$\ddot{r} = \frac{L^2\sin^2\alpha}{m^2 r^3} - g\sin\alpha\cos\alpha \tag{17}$$

Let us write the about expression in terms of height h' where h' is the height from the bottom of the cone corresponding to the radius r_0.
So, using equation (5), we have ;

$$\frac{r}{h'} = \tan\alpha = \frac{R_0}{d}$$

$$r = \frac{R_0\,h'}{d} \tag{18}$$

Using equation (15), equation (14) becomes:

$$\frac{R_0\,\ddot{h}'}{d} = \frac{L^2\,d^3\sin^2\alpha}{m^2\,R_0^3\,h'^3} - g\sin\alpha\cos\alpha \tag{19}$$

When $h' = h = constant \Rightarrow \ddot{h}' = 0$ above equation becomes;

$$\frac{L^2\,d^3\sin^2\alpha}{m^2\,R_0^3\,h^3} - g\sin\alpha\cos\alpha \tag{20}$$

Now, let $h'(t) = h + \delta(t)$ where $\delta(t)$ is a small deviation from height h.
After expanding upto first order, we have;

$$\frac{1}{h'^3} = \frac{1}{(h+\delta)^3} = \frac{1}{h^3 + 3\,h^2\,\delta} = \frac{1}{h^3\left(1+\frac{3\,\delta}{h}\right)} = \frac{1}{h^3}\left(1 - \frac{3\,\delta}{h}\right)$$

and also,

$$\ddot{r} = \frac{R_0\,\ddot{h}'}{d} \qquad [\,from\ (15)\,]$$

where

$$\ddot{h}' = \ddot{\delta}$$

Using above equations, in equation (16)

$$\frac{R_0\,\ddot{\delta}}{d} = \frac{L^2\,d^3\,\sin^2\alpha}{m^2\,R_0^3\,h^3}\left(1 - \frac{3\,\delta}{h}\right) - g\sin\alpha\cos\alpha \tag{21}$$

Now, using equation (17), above equation becomes;

$$\frac{R_0\,\ddot{\delta}}{d} = -\left(\frac{3\,L^2\,d^3\,\sin^2\alpha}{m^2\,R_0^3\,h^4}\right)\delta$$

$$\Rightarrow \ddot{\delta} = -\left(\frac{3\,L^2\,d^4\,\sin^2\alpha}{m^2\,R_0^4\,h^4}\right)\delta \tag{22}$$

So, from the above equation, the frequency of oscillation around the height h is given by,

$$\Omega = \frac{3\,L^2\,d^4\,\sin^2\alpha}{m^2\,R_0^4\,h^4}$$

(3) Double pendulum: [**10pts**] *Find the Lagrangian for the double pendulum shown in the lecture notes, and from that the Lagrange equations.*

Solution: In cartesian coordinates, the coordinates of mass m_1 are given by;

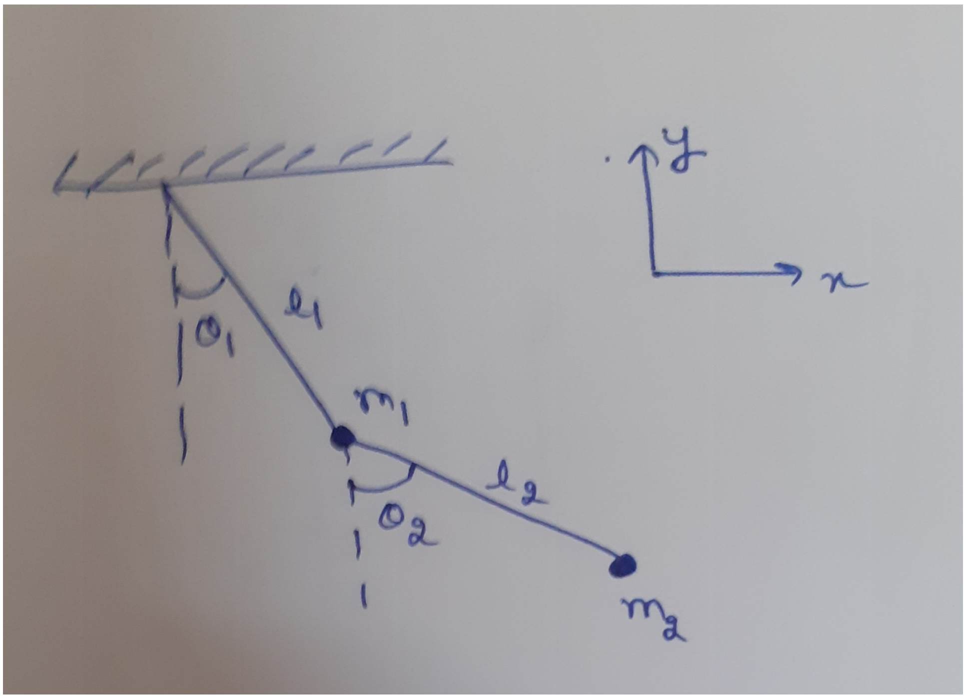
$$x_1 = l_1\sin\theta_1 \Rightarrow \dot{x_1} = l_1\cos\theta_1\,\dot{\theta_1}$$

$$y_1 = -l_1\cos\theta_1 \Rightarrow \dot{y_1} = l_1\sin\theta_1\,\dot{\theta_1}$$

and that of mass m_2 are given by;

$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \Rightarrow \dot{x_2} = l_1\cos\theta_1\,\dot{\theta_1} + l_2\cos\theta_2\,\dot{\theta_2}$$

$$y_2 = -(l_1\cos\theta_1 + l_2\cos\theta_2) \Rightarrow \dot{x_2} = l_1\sin\theta_1\,\dot{\theta_1} + l_2\sin\theta_2\,\dot{\theta_2}$$



The kinetic energy of mass m_1 is

$$T_1 = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2)$$

$$\Rightarrow T_1 = \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2$$

and the kinetic energy of mass m_2 is

$$T_2 = \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2)$$

$$\Rightarrow T_2 = \frac{1}{2}m_2(l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta_1} \dot{\theta_2}(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2))$$

$$\Rightarrow T_2 = \frac{1}{2}m_2(l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} \cos(\theta_1 - \theta_2))$$

So, the total kinetic energy T is given by;

$$T = T_1 + T_2$$

$$T = \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2}m_2(l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} \cos(\theta_1 - \theta_2)) \tag{23}$$

Now, the potential energy of mass m_1 is given by;

$$V_1 = -m_1 g\, l_1 \cos\theta_1$$

and of mass m_2 is

$$V_2 = -m_2 g\,(l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

The total potential energy is

$$V = V_1 + V_2$$

$$\Rightarrow V = -m_1 g\, l_1 \cos\theta_1 - m_2 g\,(l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$\Rightarrow V = -(m_1 + m_2) g\, l_1 \cos\theta_1 - m_2 g\, l_2 \cos\theta_2 \tag{24}$$

Hence, using (20) and (21),the Lagrangian $L = T - V$ is given by

$$L = \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2}m_2(l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} \cos(\theta_1 - \theta_2)) + (m_1 + m_2) g\, l_1 \cos\theta_1 + m_2 g\, l_2 \cos\theta_2 \tag{25}$$

The Lagrange equation is given by;
For θ_1;

$$(m_1 + m_2)\, l_1^2 \ddot{\theta_1} + m_2 l_1 l_2 \ddot{\theta_2} \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g\, l_1 \sin\theta_1 = 0$$

For θ_2;

$$m_2 l_2^2 \ddot{\theta_2} + m_2 l_1 l_2 \ddot{\theta_1} \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

(4) Bead on a spinning hoop: [10pts] *Read again example 13 of the lecture notes.*

(4a) Then implement the equation of motion using the same techniques as for assignment 1 Q4, into the same template code.

(4b) Plot the dynamics for some different initial angles $\theta(t = 0)$, for $\omega > \omega_c$, or $\omega < \omega_c$, include positions very close to the equilibrium angles θ_0 in your initial choices. Discuss your results.

Solution: *When we are below the critical frequency $\omega < \omega_c$, the bead likes to sit at the bottom of the ring and when started elsewhere oscillates over the bottom, Fig. 1. For*

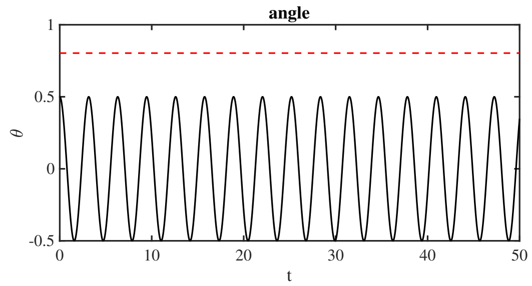


Figure 1: Below the critical frequency, the bead likes to be at the bottom.

$\omega > \omega_c$, the bottom becomes an unstable fixed point (see e.g. book by Taylor), and even if we start very close to it, with $\theta(t = 0) = 0.00001$, the bead moves far away from this point, being driven to the other fixed point at $\theta_0 = \cos^{-1}(\frac{g}{\omega^2 R})$, which happens to be stable, see Fig. 2 (left). Starting near that stable fixed point, the bead performs small harmonic oscillations around θ_0 (right).

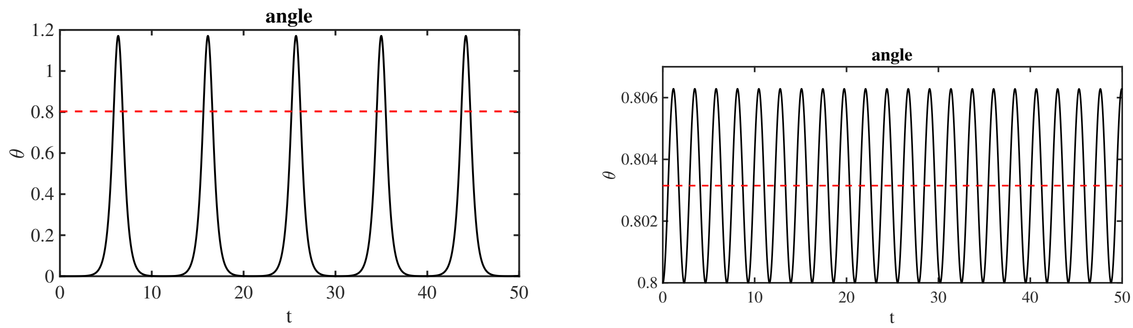


Figure 2: Starting the bead near an unstable (left) or stable (right) fixed point gives quite different motion.

(4c) A realistic bead would experience sliding friction on the hoop. Assume this is manifest in the equation of motion by an additional damping term $\ddot{\theta}(t) = -\gamma\theta(t)$, and rerun simulations from some arbitrary initial conditions. What do you find?

Solution: *In the presence of friction oscillations are damped, as expected (Fig. 3, left). This time if we start near the unstable fixed point, the bead will actually move to the stable fixed point and settle there (Fig. 3, right).*

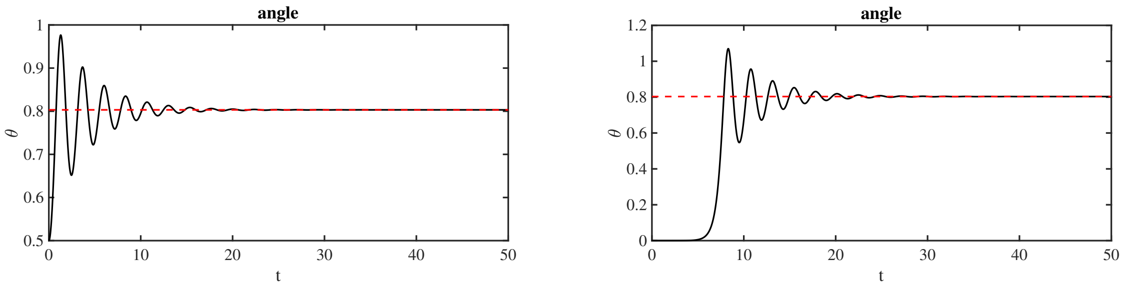


Figure 3: Bead dynamics in the presence of friction.