PHY 305, I-Semester 2020/21, Assignment 2, solution

(1) Lagrange equations: [**10pts**] A spherical pendulum is shown in Fig. 1 and described in the caption.

(1a) Considering the action of gravity on the pendulum mass, write down the Lagrangian and from that the Lagrange equation using generalized coordinates θ and φ.

(1b) Interpret the equation for φ in terms of a physical conservation law. Give the conserved quantity a name (symbol).

(1c) Try to reach an equation of the form $\ddot{\theta} = -\frac{\partial}{\partial\theta}V_{\text{eff}}(\theta)$, where $V_{\text{eff}}(\theta)$ is called the "effective potential" for θ. For this you can use the conserved quantity from (1b) to eliminate the φ dependence in the θ equation. Make a plot or drawing of $V_{\text{eff}}(\theta)$, and based on your results from (1b) and (1c) discuss which dynamics you expect for the pendulum <u>without</u> actually solving the differential equations.

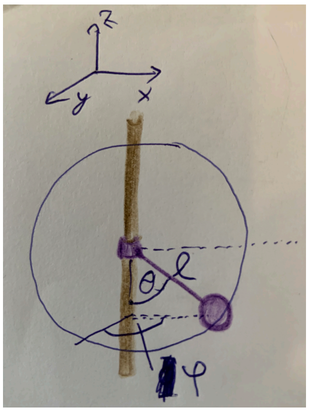

Figure 1: Sketch of spherical pendulum. A ball of mass m (violet) is attached on a rigid massless stick of length l that can somehow freely rotate in all directions. The position of the ball is thus best described in terms of spherical polar angles θ and φ. We ignore the support beam (brown) and assume all angles are possible.

Solution

1(a) *Lets assume the zero of potential energy is defined at the suspension point of the massless stick, the potential energy of the pendulum is* $V = -mglcos(\theta)$. *The kinetic energy of the pendulum with rigid stick of the length* l *can be written as:*

$$K.E. = \frac{1}{2}m[l^2\dot{\theta}^2 + l^2 sin^2(\theta)\dot{\phi}^2], \tag{1}$$

where θ *and* ϕ *are the generalized coordinates of the pendulum. To find this you start with*

$$K.E. = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2], \tag{2}$$

and then insert the constraint equations

$$
\begin{aligned}
x(t) &= l\sin(\varphi)\sin(\theta),\\
y(t) &= l\cos(\varphi)\sin(\theta),\\
z(t) &= l\cos(\theta).
\end{aligned}
\tag{3}
$$

and then use the product rule a few times and $\sin^2 + \cos^2 = 1$ *a few times.*

The Lagrangian of the pendulum can now be written as:

$$
\begin{aligned}
L &= K.E. - V\\
&= \frac{1}{2}m[l^2\dot{\theta}^2 + l^2\sin^2(\theta)\dot{\phi}^2] + mgl\cos(\theta).
\end{aligned}
\tag{4}
$$

The Lagrange equation for the generalized coordinates θ *and* ϕ *are found from:*

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0\\
\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= 0
\end{aligned}
\tag{5}
$$

and turn out as:

$$ml^2\ddot{\theta} - ml^2\sin(\theta)\cos(\theta)\dot{\phi}^2 + mgl\sin(\theta) = 0, \tag{6}$$

$$\frac{d}{dt}(ml^2\sin^2(\theta)\dot{\phi}) = 0. \tag{7}$$

1(b) *Using (7), the quantity* $L_z = ml^2\sin^2(\theta)\dot{\phi}$ *is constant in time. It turns out this is the angular momentum along the z-direction, hence the name.*
1(c) *Using the information of* $\dot{\phi}$ *from section 1(b), equation of motion for generalized coordinate (6) can be written as:*

$$
\begin{aligned}
ml^2\ddot{\theta} &= -\Bigg(- ml^2 sin(\theta)cos(\theta)\left(\frac{L_z}{ml^2 sin^2(\theta)^2}\right)^2 + mgl\sin(\theta)\Bigg),\\
\ddot{\theta} &= -\Bigg(- \frac{L_z^2 cos(\theta)}{m^2l^4\sin^3(\theta)} + \frac{g}{l}\sin(\theta)\Bigg),\\
\ddot{\theta} &= -\frac{\partial}{\partial\theta}\Bigg(\frac{1}{2}\frac{L_z^2}{m^2l^4\sin^2(\theta)} - \frac{g}{l}\cos(\theta)\Bigg),\\
\ddot{\theta} &= -\frac{\partial}{\partial\theta}(V_{eff}(\theta)),
\end{aligned}
\tag{8}
$$

where $V_{eff}(\theta) = \frac{1}{2}\frac{L_z^2}{m^2l^4\sin^2(\theta)} - \frac{g}{l}\cos(\theta)$ *is the effective potential experienced by pendulum.*

An important point here: we must substitute for L_z *into the equations of motion (6). If you substitute* L_z *for* $\dot{\phi}$ *directly into the Lagrangian (4), you will derive an equation that* *looks like the one above, but you will get a minus sign wrong! This is because Lagrange's equations are derived under the assumption that θ and ϕ are independent coordinates.*

We plot the effective potential in Fig. 2. As the Lagrangian (4) does not explicitely depend on the time coordinate (t), the total energy of the pendulum is a conserved quantity. For a given value of energy ($E = -0.5$), particle motion is restricted to the region $V_{eff} \leq E$. So from the Fig. 2 we see that the motion is pinned between two points $\theta_1 (\approx 0.09)$ and $\theta_2 (\approx 1.04)$, where θ_1 and θ_2 are the angles at which the total energy curve (red line) crosses effective potential curve (blue line). If we draw the motion of the pendulum in real space, it must therefore look something like Fig. 3, in which the bob oscillates between the two extremes: $\theta_1 \leq \theta \leq \theta_2$.

There is a stable orbit which lies between the two extremal points at $\theta = \theta_0$, corresponding to the minimum of V_{eff} (at $\theta \approx 0.3$ in Fig. 2). This occurs if we balance the angular momentum L_z and the energy E just right. We can look at small oscillations around θ_0 by expanding $\theta = \theta_0 + \delta\theta$. Substituting into the equation of motion for generalized coordniate θ (8) and ignoring higher order terms, we have

$$\begin{aligned} \delta\ddot{\theta} &\approx -\left(\frac{\partial^2 V_{eff}}{\partial \theta^2}|_{\theta=\theta_0}\right)\delta\theta \\ \delta\ddot{\theta} &\approx -\Omega^2 \delta\theta \\ \implies \delta\theta &\approx cos(\Omega t)\delta\theta, \end{aligned} \tag{9}$$

where $\Omega^2 = \frac{\partial^2 V_{eff}}{\partial \theta^2}|_{\theta=\theta_0}$ is the frequency of the oscillation. It can be seen from Eq. (9) that a small oscillation around the stable orbit behaves like a simple harmonic motion.

(2) Variational TISE: [5pts] Consider a quantum particle in 1D of mass m and with potential energy $V(x)$. The quantum mechanical expectation value of energy, or "energy functional" is

$$H = \int_{-\infty}^{\infty} dx \;\; \Psi^*(x) \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right] \Psi(x). \tag{10}$$

Show that the time-independent Schrödinger equation follows as Euler-Lagrange equation from finding a stationary solution of the functional $\tilde{H} = H - E \int_{-\infty}^{\infty} dx |\Psi(x)|^2$, where E initially is only a Lagrange multiplier[1].

Hints: (i) For the calculation, treat $\Psi(x)$ and $\Psi^*(x)$ as independent functions and form the EL-eqns wrt. $\Psi^*(x)$.

Solution: *We can write the complete functional as*

$$\begin{aligned} \tilde{H} &= \int_{-\infty}^{\infty} dx \left[\Psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\Psi(x) + \Psi^*(x)V(x)\Psi(x) - E\,\Psi^*(x)\Psi(x)\right] \\ &\equiv \int_{-\infty}^{\infty} dx \;\; f[\Psi^*(x), \Psi^*(x)', \Psi(x), \Psi(x)'] \end{aligned} \tag{11}$$

[1]We will explain this concept in the TA class, you don't need to know what it means for the solution.

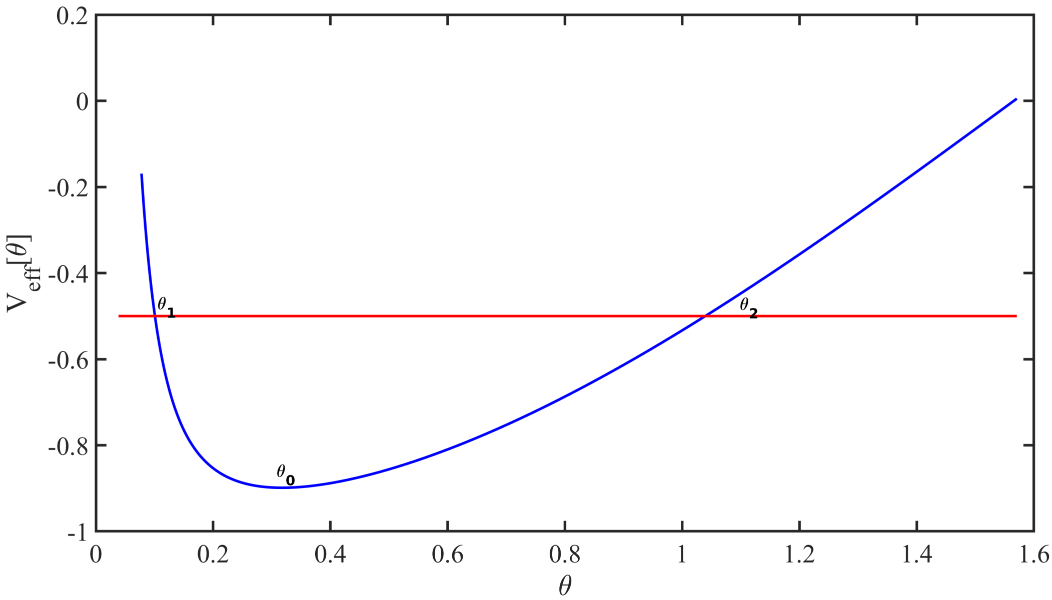

Figure 2: Effective potential (V_{eff}) vs θ is shown here with blue line for $Lz = 0.1, m = 1, l = 1$, and $g = 1$. Total energy of the pendulum is shown for $E = -0.5$ with red line.

To find the stationary solution for the functional we use the Euler Lagrange equation which reads here

$$\frac{\partial f}{\partial \Psi^*(x)} - \frac{d}{dx}\frac{\partial f}{\partial \Psi^*(x)'} = 0 \tag{12}$$

$$+V(x)\Psi(x) - E\Psi(x) - \frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} = 0 \tag{13}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x). \tag{14}$$

which is the TISE we were supposed to find. [Note: if you wanted the functional to be more symmetric in Ψ versus Ψ^* you could have first moved a spatial derivative onto Ψ^* using integration by parts and would have found the same answer]

(3) Fastest Slide: You are constructing a water-park, and want to design a slide as shown in such that your customers reach from the top to the end of the slide in the shortest time. Assuming any effect of the water is negligible, excepting making sure there is no friction, so that people slide under gravity only, use the calculus of variations to find the shape of the slide.

(3a) [4pts] First show the <u>Beltrami identity</u>: If a functional to be minimized does not depend explicitly on x, i.e.

$$S = \int_{x_1}^{x_2} dx f[y(x), y'(x)] \tag{15}$$

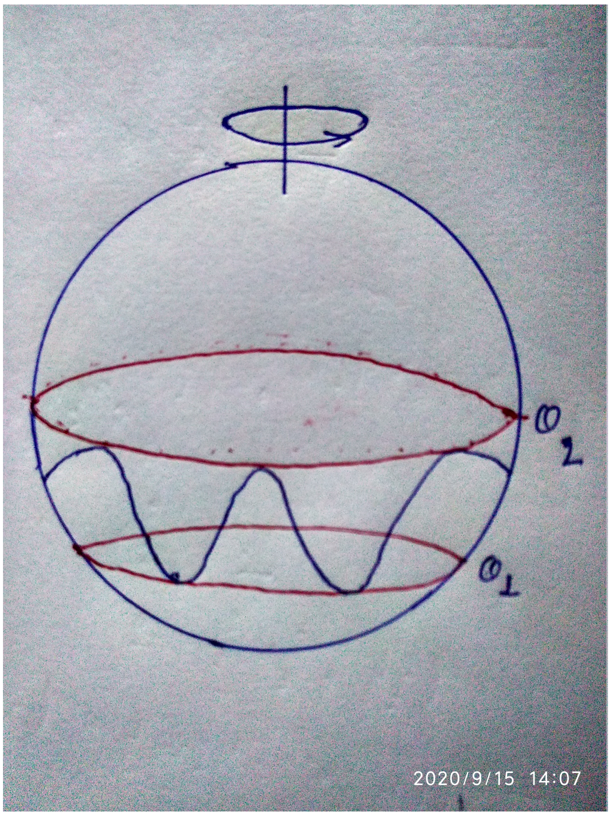

Figure 3: Motion of pendulum is shown between $\theta_1 \leq \theta \leq \theta_2$.

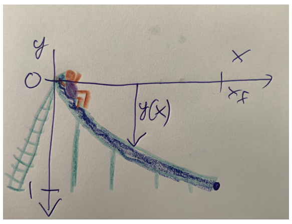

Figure 4: Waterslide, of starting height $0 = y(0)$ and length x_f. The slide profile is given by the function $y(x)$.

instead of lecture Eq. (2.6), we can use an alternative form of the Euler-Lagrange Eq. (2.12) to find the solution, namely:

$$f - y' \frac{\partial}{\partial y'} f = const. \tag{16}$$

(3b) [3pts] Show that the total time taken (and hence the functional to be minimized), is

$$T = \frac{1}{\sqrt{2g}} \int_{x=0}^{x_f} \sqrt{\frac{y'(x) + 1}{y(x)}} dx, \tag{17}$$

where g is the acceleration due to gravity at earth's surface.

(3c) [5pts] Using Eq. (16), write down the differential equation for the slide-profile $y(x)$ that minimizes (17).

(3d) [3pts] Solve that equation to show that we can parametrically write the slide profile as

$$x = \frac{2s - \sin(2s)}{1 - \cos(2s_f)}, \tag{18}$$

$$y = \frac{1 - \cos(2s)}{1 - \cos(2s_f)}, \tag{19}$$

for $0 \leq s \leq s_f$, where s_f is a solution of $x_f = \frac{2s_f - \sin(2s_f)}{1 - \cos(2s_f)}$. Depending on your method of solution you might end up with a slightly different parametrisation, you can check if it gives the same curve, by plotting it together with (19) e.g. using the script in Q4. We also shall use that script to analyze your answer.

Solution:

3(a) *If we write the EL equation for the functional,*

$$\frac{\partial f}{dy} - \frac{d}{dx}\frac{\partial f}{dy'} = 0. \tag{20}$$

If we multiply the equation with y' we can write,

$$y'\frac{\partial f}{dy} = y'\frac{d}{dx}\frac{\partial f}{dy'} \tag{21}$$

Using chain rule we get,

$$\frac{df}{dx} = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial x}, \tag{22}$$

$$\Rightarrow \frac{\partial f}{\partial y}y' = \frac{df}{dx} - \frac{\partial f}{\partial y'}y'' - \frac{\partial f}{\partial x}. \tag{23}$$

Putting (23) in (21),

$$\frac{df}{dx} - \frac{\partial f}{\partial y'}y'' - \frac{\partial f}{\partial x} - y'\frac{d}{dx}\frac{\partial f}{dy'} = 0. \tag{24}$$

By product rule the last term in (24) can be written as,

$$y'\frac{d}{dx}\frac{\partial f}{dy'} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}y'\right) - \frac{\partial f}{\partial y'}y''. \tag{25}$$

Rearranging the equation will give,

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} \tag{26}$$

Now since the functional does not explicitly depend on x we can say,

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0, \tag{27}$$

$$\Rightarrow\ f - y'\frac{\partial f}{\partial y'} = const. \tag{28}$$

3(b) *We see in Fig.(4) a small change in distance s from the origin is given by* $\delta f^2 = \delta x^2 + \delta y^2$. *Hence,*

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \tag{29}$$

$$= \left(\frac{dx}{dt}\right)^2\left[1 + \left(\frac{dy}{dt}\right)^2 \Big/ \left(\frac{dx}{dt}\right)^2\right], \tag{30}$$

$$= \left(\frac{dx}{dt}\right)^2\left(1 + (y')^2\right) \tag{31}$$

Now energy is given by,

$$E = \frac{1}{2}mv^2 - mgy \tag{32}$$

Since initial height and velocity is 0 initial energy $E = 0$. *Hence we can write,*

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 - mgy = 0, \tag{33}$$

$$\Rightarrow\ \frac{1}{2}m\left(\frac{dx}{dt}\right)^2\left(1 + (y')^2\right) - mgy = 0 \tag{34}$$

Rearrangement will give,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2gy}{1 + (y')^2}. \tag{35}$$

Now time of passage from x=0 t0 $x = x_f$ *is given by,*

$$T = \int_0^T dt = \int_0^{x_f} \frac{1}{dx/dt}dx = \int_0^{x_f}\sqrt{\frac{1 + (y')^2}{2gy}}\,dx. \tag{36}$$

Hence the total time taken by the slide is given by,

$$T = \frac{1}{\sqrt{2g}}\int_{x=0}^{x_f}\sqrt{\frac{y'(x)^2 + 1}{y(x)}}dx \tag{37}$$

3(c) *The integrand we need to minimize in (37) is,*

$$F(y, y') = \frac{1}{\sqrt{2g}} \sqrt{\frac{y'(x)^2 + 1}{y(x)}}. \tag{38}$$

We use Eq. (16) to obtain,

$$\frac{1}{\sqrt{2g}} \frac{(y')^2}{\sqrt{y(1 + (y')^2)}} - \frac{1}{\sqrt{2g}} \sqrt{\frac{(y')^2 + 1}{y}} = const. \tag{39}$$

Simplyfying this gives,

$$\frac{1}{\sqrt{y(1 + (y')^2)}} = \frac{1}{c}, \tag{40}$$

where $\sqrt{2g}$ is absorbed into the constant c. Hence we will get,

$$y'(x) = \pm \sqrt{\frac{c^2 - y}{y}}. \tag{41}$$

3(d) *Eq. (41) can be written as,*

$$\int dx = \int \sqrt{\frac{y}{c^2 - y}}\, dy. \tag{42}$$

To solve this integral we can substitute $y = c^2 \sin^2(s)$. Hence,

$$\int \sqrt{\frac{y}{c^2 - y}}\, dy = 2c^2 \int \sin(s) \cos(s) \sqrt{\frac{c^2 \sin^2(s)}{c^2 - c^2 \sin^2(s)}}, \tag{43}$$

$$\Rightarrow \int \sqrt{\frac{y}{c^2 - y}}\, dy = \frac{1}{2} c^2 (2s - \sin(2s)) + d, \tag{44}$$

where d is a constant. Hence we have,

$$x = \frac{1}{2} c^2 (2s - \sin(2s)) + d \;\; and \;\; y = c^2 \sin^2(s) = \frac{1}{2} c^2 \big(1 - \cos(2s)\big). \tag{45}$$

When s=0, the equations give x=d and y=0. But since at y(0)=0 we can write d=0. Now when $s = s_f \Rightarrow x = x_f$, then

$$x_f = \frac{1}{2} c^2 \big(2s_f - \sin(2s_f)\big) = \frac{2s_f - \sin(2s_f)}{1 - \cos(2s_f)}, \tag{46}$$

$$\Rightarrow c^2 = \frac{2}{1 - \cos(2s_f)}. \tag{47}$$

Therefore parametric equations will be,

$$x = \frac{2s - \sin(2s)}{1 - \cos(2s_f)}, \tag{48}$$

$$y = \frac{1 - \cos(2s)}{1 - \cos(2s_f)}, \tag{49}$$

(4) Computational question: Fastest Slide again: [**10pts**] We now try to get the results from Q3 with the computer instead. For nicer plots, we slightly change coordinates, using $\tilde{y}(x) = h - y(x)$, where $y(x)$ is the y-coordinate from Q3. From the computational point of view, the variational problem falls into the class of "optimisation problems". You have a function $f(p_1, p_2, p_3, \cdots)$ that depends on a large number of parameters p_k and want to optimise it. In our case, $f = T$ from Eq. (17) and the input parameters are the function values $y_k = y(x_k)$ at a set of discretely sampled position points x_k reaching from 0 to x_f.

(4a) The code `Assignment2_program_draft_v4.m` is set up to perform this optimisation on one such discretely sampled function. You need to only insert items at the points XXX. Any numerical optimisation requires an initial "guess" $y(x)_{\text{guess}}$. Implement that in the first XXX. The other ones need to be filled such that the functional correctly evaluates (17). Test your implementation by running the code, loading the output and using `Assignment2_plot_slide_v2.m` to compare the output with the solution from (19). Discuss what you see.

Solution: *We see in Fig. 5, that after a relatively large number of iterations (here `options.MaxIterations=20000` and `options.MaxFunctionEvaluations= 1000000`, the numerical solution almost approaches the analytical one. There is a residual discrepancy and this problem is surprisingly challenging for the computer.*

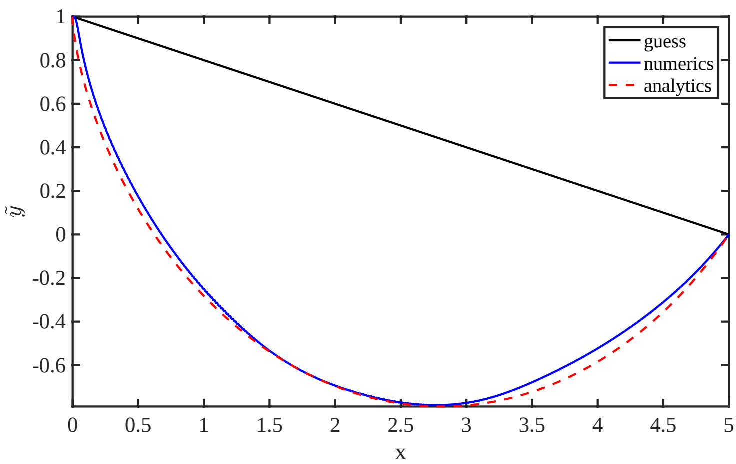

Figure 5: Comparison of analytical solution, numerical solution and initial guess.

(4b) What the optimizer practically does is somewhat akin to the figure above Eq. (2.8)

in the lecture: It takes your $y(x)_{\text{guess}}$ and then varies the path in the vicinity by a small offset $\eta(x)$. If the offset reduces T, it keeps the change and tries again. In order to visualize this, produce a set of plots for quite small parameter `options.MaxIterations`= 5,10,20,50,100 etc. This aborts the optmisation after relatively few attempts even if the result is not yet good. See how it progressively becomes better for more iterations.
Solution: *We see in Fig. 6 how the optimizer slowly approaches something resembling the real solution, but for these low numbers of iterations, we are still far away from the true solution.*

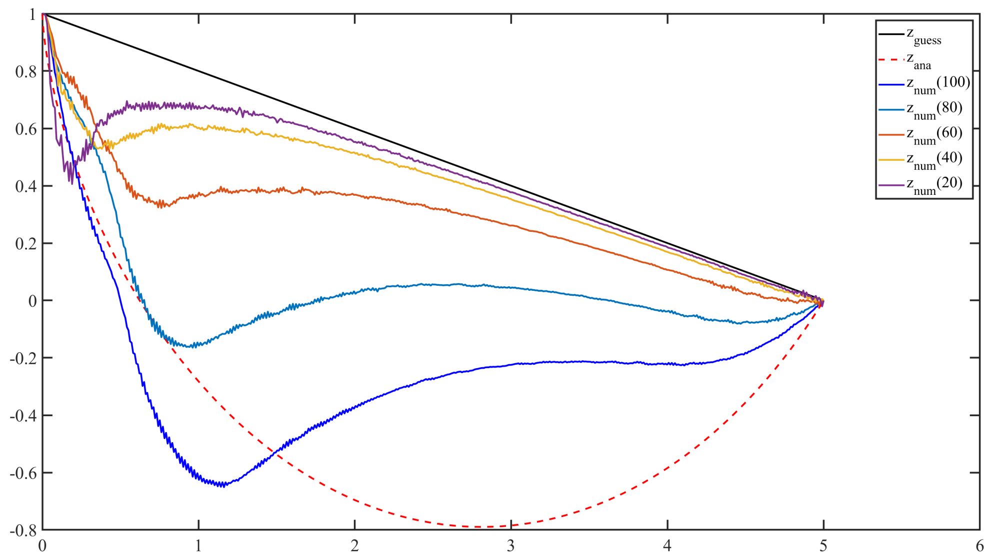

Figure 6: A comparison of $N_{it} = 20, 40, 60, 80, 100$ with the analytical calculation is shown in this figure, where N_{it} is the number of iterations.

(4c) Now vary the parameter x_f to look at a 2-3 differently shaped slides. Discuss your results. *For plotting these, you have to also adjust x_f in the file* `phi_funct.m`.
Solution: *Slides for x_f = 2, 5 and 10 are compared in Fig. 7. We see that the longer the slide, the more prominent the counter-intuitive upwards slope at the end, in order to exploit the longer horizontal acceleration phase provided by going to negative heights $\tilde{y}$.*

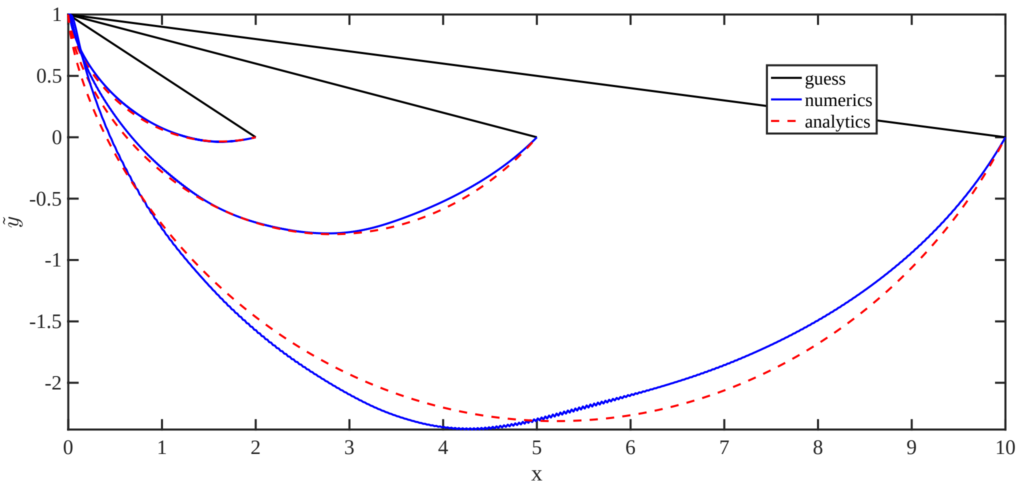

Figure 7: Slides for different values with numerical parameters as in Fig. 5.