PHY 305, I-Semester 2020/21, Assignment 1 Solution

(1) Block sliding on a wedge: A pink cube of mass m slides on a wedge without friction as shown in Fig. 2. The wedge itself has mass M and angle α and is in turn sliding on a frictionless floor. All are at rest at $t = 0$. If the cube was initially at a height h, at what time will it reach the bottom? Solve the problem using Newtonian mechanics.

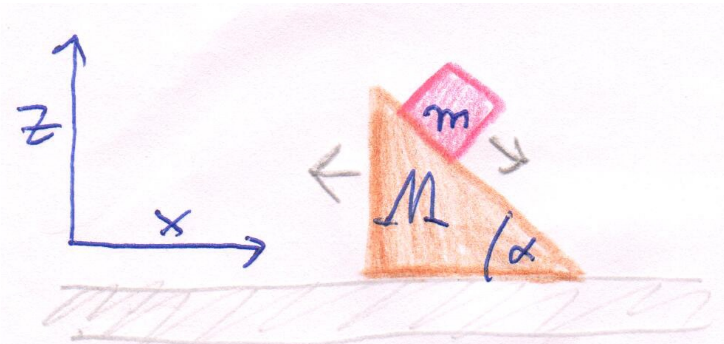

Figure 1: Cube, sliding on wedge, where the wedge is sliding on the floor.

Solution: Since the inclined plane frictionless the only force exerted on the cube by the inclined plane is the constraint force in given by N. And lets consider the cube moves with a acceleration $\mathbf{a}$ *and the inclined plane move with an acceleration* $\mathbf{A}$ *in an inertial frame of reference. We will have only* x *component of the acceleration* $\mathbf{A}$ *(hence its magnitude*

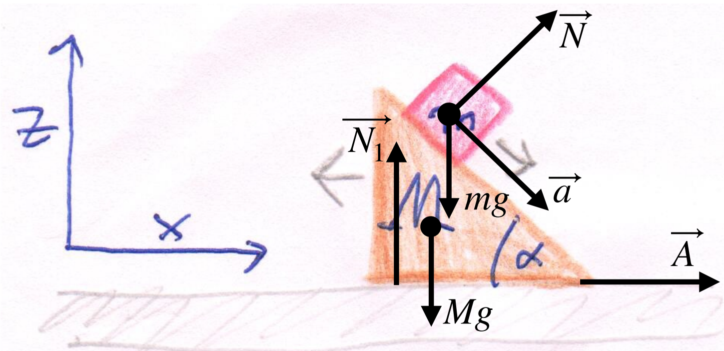

Figure 2: Cube, sliding on wedge, forces acting on the cube.

is represented as A_x*). Let the force acting on the cube be*

$$\mathbf{F}_C = m(\mathbf{a} + \mathbf{A}). \tag{1}$$

$\mathbf{F}_C$ *has parallel and perpendicular components to the slope of the inclined plane and can be written as* $\mathbf{F}_C = \mathbf{F}_{C\,,\,||} + \mathbf{F}_{C\,,\,\perp}$.

$$\mathbf{F}_{C\,,\,||} = mg\,sin(\alpha) = ma_{||} + mA_x\,cos(\alpha), \tag{2}$$

$$\mathbf{F}_{C\,,\,\perp} = N - mg\,cos(\alpha) = ma_{\perp} + mA_x\,sin(\alpha), \tag{3}$$

where $ma_{||}+mA_x cos(\alpha)$ *is the parallel component of RHS of Eq.(1) and* $ma_{\perp}+mA_x sin(\alpha)$ *is the perpendicular component. Since the constraint is present* $a_{\perp}=0$*. Hence constraint force will be,*

$$N = mg\, cos(\alpha) + mA_x\, sin(\alpha). \tag{4}$$

Now looking at the net force acting on the inclined plane. According to Newton's Third law there will be a equal and opposite force acting inclined plane by the cube which is $-\mathbf{N}$ *and another constrain force acted by the floor on the inclined plain which is written as* $\mathbf{N}_1 = N_1\,\hat{j}$ *(in y direction). Hence we can write,*

$$\mathbf{F}_{IP} = -\mathbf{N} + \mathbf{N}_1 - Mg\,\hat{j} = M\mathbf{A}. \tag{5}$$

Here $N_1 - Mg$ *cancels the downward component of* $-\mathbf{N}$ *resulting to,*

$$\mathbf{N}_1 - Mg = -N\, cos(\alpha). \tag{6}$$

Also the horizontal component of $-\mathbf{N}_1$ *equals to the acceleration of the inclined plain,*

$$MA_x = -N\, sin(\alpha). \tag{7}$$

Using Eqs. (7) and (4) we can write acceleration of the inclined plain as,

$$A_x = -g\left(\frac{sin(\alpha)\, cos(\alpha)}{sin^2(\alpha) + \frac{M}{m}}\right). \tag{8}$$

From Eq. (2) the acceleration of the cube along the plane $a_{||}$ *can be obtained as,*

$$a_{||} = g\, sin(\alpha) - A_x\, cos(\alpha). \tag{9}$$

The acceleration of the cube down the slope is given by,

$$a_{||} = g\, sin(\alpha)\left(\frac{M+m}{M+m\, sin^2(\alpha)}\right). \tag{10}$$

The cube which was initially at a height h will reach bottom in a time t given by solution to $\frac{h}{sin(\alpha)} = a_{||}\,\frac{t^2}{2}$*,which gives* $t=\sqrt{2h/a_{||}\, sin(\alpha)}$*. Hence,*

$$t = \sqrt{\frac{2h}{g\, sin^2(\alpha)\left(\frac{M+m}{M+m\, sin^2(\alpha)}\right)}} \tag{11}$$

(2) Two-dimensional harmonic oscillator: *A two-dimensional harmonic oscillator of mass m is one which has two-degrees of freedom x and y, and with a potential energy*

$$V_{pot}(x,y) = \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right). \tag{12}$$

(2a) Using the definition of the gradient, write down Newton's equation for this oscillator explicitly in terms of x, y.
Solution: Newton's equation is $m\ddot{\mathbf{r}} = \mathbf{F} = -\boldsymbol{\nabla} V_{pot}(x, y)$. Using Eq. (1.13) of the lecture for 2D, we evaluate

$$\boldsymbol{\nabla} V_{pot}(x, y) = \begin{bmatrix} \frac{\partial}{\partial x} V_{pot}(x, y) \\ \frac{\partial}{\partial y} V_{pot}(x, y) \end{bmatrix} = \begin{bmatrix} m\omega_x^2 x \\ m\omega_y^2 y \end{bmatrix} \tag{13}$$

which yields for Newton's equation

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = - \begin{bmatrix} m\omega_x^2 x \\ m\omega_y^2 y \end{bmatrix}, \tag{14}$$

or without vector notation $\ddot{x} = -\omega_x^2 x$ and $\ddot{y} = -\omega_y^2 y$.

(2b) From your result in (a) and your knowledge of the 1D harmonic oscillator, find all conserved quantities, discuss how the dynamics will look like. What is the difference between this 2D harmonic oscillator and two separate 1D harmonic oscillators?
Solution: Let's answer the last question first: from the equation we can see that there is <u>no</u> difference between the equation of motion of the x and y components of a 2D harmonic oscillator compared to the case where these coordinates would describe two completely different oscillators, say $x = r_1$, $y = r_2$. We say that the two dimensions completely <u>de-couple</u>. We can thus use our knowledge of a 1D oscillator to directly state that the conserved quantities will be $E_x = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega_x^2 x^2$ and $E_y = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} m\omega_y^2 y^2$ the (total) energies associated with the motion in the x and y direction, and the general solution will be $x(t) = C_1 \sin(\omega_x t) + C_2 \cos(\omega_x t)$ and $y(t) = C_3 \sin(\omega_y t) + C_4 \cos(\omega_y t)$, where the C_k are determined by the initial conditions.

(2c) Find the time-averaged kinetic energy and potential energy separately for the motion related to x. How are they related?
Solution: For a period $T = 2\pi/\omega_x$, using the result above,

$$\begin{aligned} \bar{E}_{kin} &= \frac{1}{2} m \int_0^T \dot{x}(t)^2 dt/T = \frac{m}{2T} \int_0^T (C_1\omega_x \cos(\omega_x t) - C_2\omega_x \sin(\omega_x t))^2 dt \\ &= \frac{m\omega_x^2}{2T} \int_0^T [C_1^2 \cos^2(\omega_x t) - 2C_1C_2 \cos(\omega_x t) \sin(\omega_x t) + C_2^2 \sin^2(\omega_x t)] dt \end{aligned} \tag{15}$$

Using a drawing of $\sin$ and $\cos$ we can immediately tell that the integral of their product over one period is zero. Separate integration tell us that $\int_0^T \sin(\omega_x t)^2/T = \int_0^T \cos(\omega_x t)^2/T = 1/2$ (this result is needed so frequently, it is worth remembering). Thus $\bar{E}_{kin} = \frac{m\omega_x}{4}(C_1^2 + C_2^2)$.

Doing the same for the potential energy average:

$$\bar{E}_{pot} = \frac{1}{2}m\omega_x^2 \int_0^T x(t)^2 dt/T$$
$$= \frac{m\omega_x^2}{2T}\int_0^T [C_1^2 \sin^2(\omega_x t) + 2C_1C_2 \sin(\omega_x t)\cos(\omega_x t) + C_2^2 \cos^2(\omega_x t)]dt \tag{16}$$

Using the same arguments as before we reach $\bar{E}_{pot} = \frac{m\omega_x}{4}(C_1^2 + C_2^2)$. *Thus the average kinetic and average potential energy of the harmonic oscillator are the same.*

(3) Line integrals *The harmonic oscillator from the previous questions is being moved from position* $\mathbf{r}_1 = (x_1, y_1) = (1, 0)$ *to position* $\mathbf{r}_2 = (x_2, y_2) = (0, 3)$ *in a straight line in 2D. Calculate the work exerted on the oscillator using the explicit line integral over the force in two dimensions. Which would be a much simpler way to arrive at the same answer? Solution: We follow the definition of the line integral over a vector function in the lecture. We first need a parametrisation of the chosen path, which can be for example* $\mathbf{s}(t) = \mathbf{r}_1 + (\mathbf{r}_2 - \mathbf{r}_1)t$ *for* $0 \le t \le 1$, *or in vector form*

$$\mathbf{s}(t) = \begin{bmatrix} 1-t \\ 3t \end{bmatrix}, \qquad \mathbf{ds}(t) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \tag{17}$$

To find the work W_{12}, *we now plug together Eq. (17) and the gradient Eq. (14) using Eq. (1.9) and Eq. (1.10) of the lecture to find;*

$$W_{12} = \int_1^2 \mathbf{F}\cdot\mathbf{ds} = \int_0^1 [-\boldsymbol{\nabla}V(\mathbf{s}(t))]\cdot\mathbf{ds}(\mathbf{t})dt = -\int_0^1 \begin{bmatrix} m\omega_x^2 x(t) \\ m\omega_y^2 y(t) \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 3 \end{bmatrix} dt$$
$$= -\int_0^1 [-m\omega_x^2(1-t) + 3m\omega_y^2(3t)]dt = \left[m\omega_x^2(t - t^2/2) - 3m\omega_y^2(3t^2/2)\right]_0^1$$
$$= \frac{1}{2}m\omega_x^2(1)^2 - \frac{1}{2}m\omega_y^2(3)^2. \tag{18}$$

We recognise this as the difference between the initial and final potential energies $W_{12} = V(\mathbf{r}_2) - V(\mathbf{r}_1)$, *which would of course have been the much simpler way to arrive at the same result.*

(4) Computational question: harmonic oscillator, damping driving and modulation *Newton's equation for a point mass* m *in one-dimension subject to a force* $F(t)$ *is* $m\ddot{r} = F(t)$.

(4a) Computational algorithms typically need us to convert second order differential equations in time into a system of first order differential equations in time. By using both, the position r *and the velocity* v *as variables, do this conversion, i.e. find a coupled system of first order differential equations that is equivalent to Newton's equation.*

Solution: We can simply write

$$\dot{\mathbf{x}} = \mathbf{v},$$
$$\dot{\mathbf{v}} = F(t)/m. \tag{19}$$

(4b) Write these equations for the case of a 1D simple harmonic oscillator, where $F_k = -kx$ and implement them in the template file `Assignment1_program_draft_v1.m`. Solve the equations for a couple of different choices of parameters, and analyse your result using `Assignment1_plot_oscillator_v1.m` and `Assignment1_plot_phasespace_v1.m`, discuss.

Solution : The classical harmonic oscillator in 1D can be written as ,

$$\ddot{\mathbf{x}} = -k\mathbf{x}/m \tag{20}$$

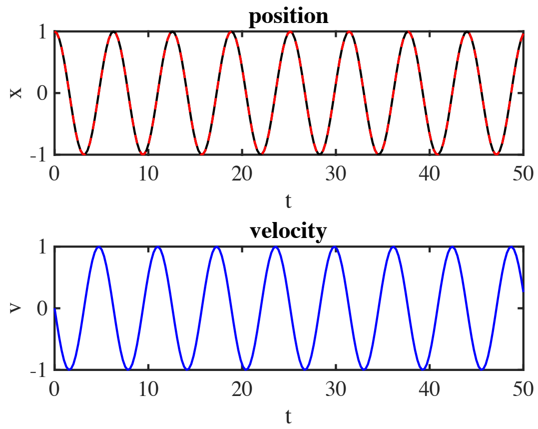

Figure 3: Position and velocity vs time plot for a harmonic oscillator in the absence of driving force.

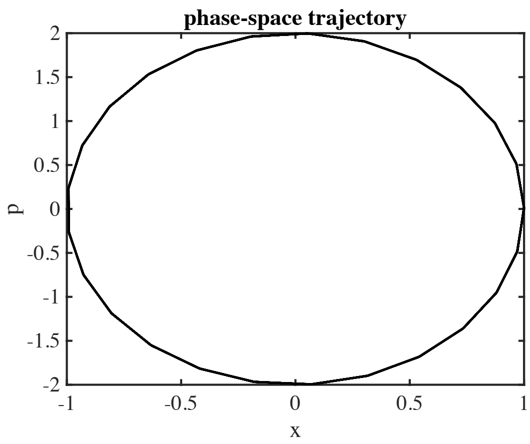

Figure 4: Phasespace diagram plot for a harmonic oscillator in the absence of driving force.

Since there is no external external driving or damping force the oscillator keeps on oscillating with the same amplitude with the time and follows an elliptical path in phasespace.

(4c) Now we extend the calculations to add a friction force $F_f = -\gamma v$ and a driving force $F_d = F_0 \sin(\omega t)$. Again analyze your result and compare it with solutions for the damped, driven harmonic oscillator that you can find in textbooks or the internet. (Use some arbitrary dimensionless units throughout question 4, e.g. mainly pick parameters in the range $[0, 5]$).

Solution :

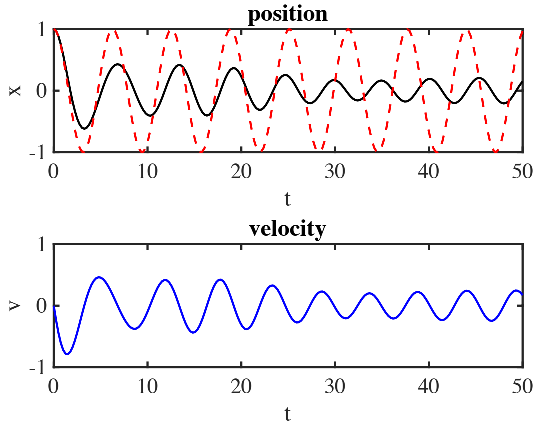

Figure 5: Position and velocity vs time plot for a harmonic oscillator in the presence of the driving force.

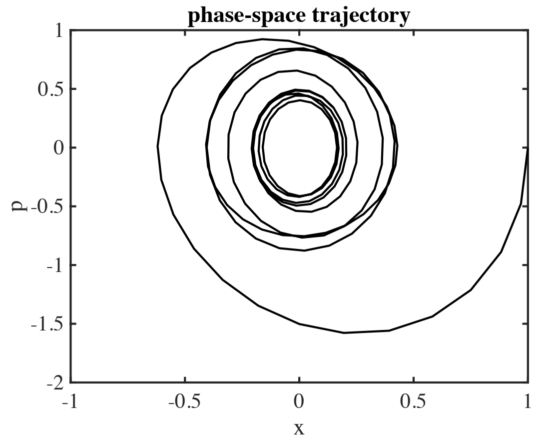

Figure 6: Phasespace diagram plot for a harmonic oscillator in the presence of the driving force.