Week 9

PHY 305 Classical Mechanics
Instructor: Sebastian Wüster, IISER Bhopal, 2020

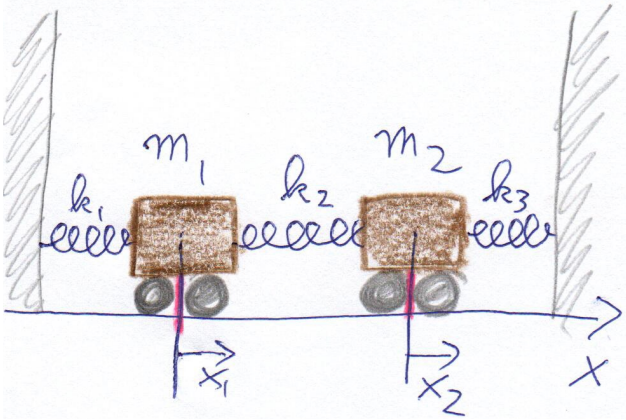
These notes are provided for the students of the class above only. There is no guarantee for correctness, please contact me if you spot a mistake.

3.6 Coupled oscillators

At the beginning of section 3.1.1 we had seen that you can imagine a rigid body as composed of many mass points with fixed distances between them, and motivated this also by the idea that after all the rigid body really is composed of bound molecules. However the forces keeping the molecules together, will not rigidly fix their distances, but instead allow them vibrations or oscillations around equilibrium positions. However neighboring molecules will strongly affect one another. In reality this thus turns into a system of coupled oscillators which we study now more closely.

3.6.1 Two coupled oscillators

Let's first see what happens in a simple example, which is somewhat unrealistic but chosen for easy concepts: Two carts can move without friction on a rail, and are attached in between two walls via springs as shown in the figure below.



left: The masses of the carts are m_1 and m_2, the spring constants of the three springs k_1, k_2, k_3 as shown. We take coordinates where the original position of cart k is x_n, and assume that for $x_1 = x_2 = 0$, which denote the equilibrium (pink) position of the carts, all three springs are in their natural position (neither stretched nor compressed).

In the absence of the middle spring with k_2, each of these carts just forms a simple harmonic oscillator with known frequency $\omega_n = \sqrt{k_n/m_n}$. However due to the middle spring, you cannot shift cart 1 without exerting a pulling or pushing force onto cart 2, so the two oscillators are now coupled.

We set up Newton's (or equivalently Lagrange's) equations, and find:

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1), \\ m_2\ddot{x}_2 &= -k_3x_2 - k_2(x_2 - x_1). \end{aligned} \tag{3.58}$$

As expected since the oscillators are coupled, the equation for $\ddot{x}_1$ depends on x_2 and vice versa. We can solve (3.58) by adding or subtracting the two equations, but we will soon encounter more complicated coupled systems (with more than two variables), so let us instead directly learn a more powerful approach. Let us form a vector of positions $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ and write (3.58) as

$$\ddot{\mathbf{x}}(t) = \tilde{M}\mathbf{x}(t), \text{ with matrix } \tilde{M} = \begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_3+k_2}{m_2} \end{bmatrix}. \tag{3.59}$$

Motivated by the need to solve matrix-differential equations of this kind, we look at:

Coupled system of linear differential equations: Consider the system of differential equations with constant coefficients

$$\begin{aligned} \frac{d}{dt}f_1(t) &= m_{11}f_1(t) + m_{12}f_2(t) + \cdots + m_{1N}f_N(t), \\ \frac{d}{dt}f_2(t) &= m_{21}f_1(t) + m_{22}f_2(t) + \cdots + m_{2N}f_N(t), \\ &\cdots \\ \frac{d}{dt}f_N(t) &= m_{N1}f_1(t) + m_{N2}f_2(t) + \cdots + m_{NN}f_N(t). \end{aligned} \tag{3.60}$$

for functions $f_k(t)$. To warmup we look at an equation of first order in time initially (unlike (3.59)). Importantly the coefficients m_{kn} do <u>not</u> depend on t nor on the f_k.
We can write this in matrix form as

$$\frac{d}{dt}\mathbf{f}(t) = M\mathbf{f}(t), \tag{3.61}$$

where we have grouped all functions into an N-component vector $\mathbf{f}(t)^T = [f_1(t), f_2(t), \ldots f_N(t)]$ and all coefficients m_{kn} into the $N \times N$ matrix M. Let us assume we have found all N eigenvalues λ_ℓ and eigenvectors $\mathbf{v}_\ell$ of M according to

$$M\mathbf{v}_\ell = \lambda_\ell \mathbf{v}_\ell. \tag{3.62}$$

We can then show that the general solution of this system is given by

$$\mathbf{f}(t) = \sum_\ell c_\ell \mathbf{v}_\ell e^{\lambda_\ell t}, \tag{3.63}$$

where c_ℓ are a set of coefficients determined by the initial conditions $\mathbf{f}(0)$.

- That it is a solution you can proof yourself in 3 lines by inserting (3.63) into (3.61) and then using (3.62). For the the statement that it is also all solutions, we refer to math courses.
- You may think of t as time, which is the case for which we will now use this result, but it can of course be any variable.

Second order system of linear differential equations: We can now use the same trick also for a system of second order in time such as (3.59):

$$\frac{d^2}{dt^2}\mathbf{f}(t) = M\mathbf{f}(t). \tag{3.64}$$

If all eigenvalues λ_ℓ of the matrix M are non-zero, this has a solution

$$\mathbf{f}(t) = \sum_\ell \left[c_\ell^+ \mathbf{v}_\ell e^{+\sqrt{\lambda_\ell}t} + c_\ell^- \mathbf{v}_\ell e^{-\sqrt{\lambda_\ell}t}\right]. \tag{3.65}$$

- We shall require only the case $\lambda_\ell \neq 0$ here. For eigenvalues λ_ℓ you instead add a contribution $\mathbf{v}_\ell(c_\ell^0 + c_\ell^1 t)$ to the sum.
- When you are reading the books, you shall find that I have deviated in my choice of how to present the solutions for the coupled differential equation systems. I find the presented way more straightforward than the ones from the books, but shall comment on the latter near Eq. (3.85) later. If one of the readers can tell me a reason for introducing "generalized eigenvalue problems" to solve the coupled oscillator, which is better than "all the books do it", they get a cookie.

We can now apply Eq. (3.65) to Eq. (3.59). The general case, despite there being only two oscillators, is still a bit messy, so we look at two instructive cases:

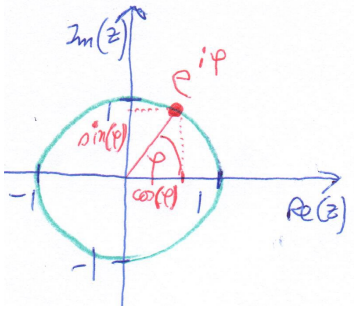
Equal masses and springs:

First we look at carts of equal mass $m_1 = m_2 = m$ coupled by three identical springs $k_1 = k_2 = k_3 = k$. Then $\tilde{M}$ in Eq. (3.59) becomes

$$\tilde{M} = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix}, \tag{3.66}$$

which has eigenvalues $\lambda_0 = -k/m$ and $\lambda_1 = -3k/m$, with corresponding normalized eigenvectors $\mathbf{v}_0 = [1,1]^T/\sqrt{2}$ and $\mathbf{v}_1 = [1,-1]^T/\sqrt{2}$. Since k and m are positive, the λ_ℓ are negative, so when taking the square root in (3.65), we find complex numbers, e.g. $\sqrt{\lambda_0} = i\sqrt{k/m}$ etc. The solution thus contains

Complex exponential function:



left:

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t). \tag{3.67}$$

For a function $z = e^{i\varphi}$, we have $|z| = 1$, so the complex number sits always on the unit circle in the complex plane. The angle φ is called the "phase" of that number.

Link to cos and sin: We can invert Eq. (3.67) to find

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}),$$
$$\sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}). \tag{3.68}$$

Using these functions, we can write the solution for the two coupled carts in Eq. (3.59) explicitly as

$$\ddot{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(c_0^+ e^{i\omega_0 t} + c_0^- e^{-i\omega_0 t}\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(c_1^+ e^{i\omega_1 t} + c_1^- e^{-i\omega_1 t}\right), \tag{3.69}$$

where we defined $\omega_0 = \sqrt{k/m}$ and $\omega_1 = \sqrt{3k/m}$.

- As stated before, the (complex) coefficients c are determined by the initial conditions $\mathbf{x}(t=0)$ and $\dot{\mathbf{x}}(t=0)$.
- It might be on first sight confusing that complex numbers enter the solution. However this is making our calculations easier, and represents no problem. To see the latter, convince yourself of the following important points: (i) If $\mathbf{x}(t=0)$ is real, then according to (3.59) it will remain real at all times, since the matrix $\tilde{M}$ is real. (ii) If a complex function $\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t)$ with real $\mathbf{x}$, $\mathbf{y}$ satisfies Eq. (3.59), then also the real part $\mathbf{x}$ and imaginary part $\mathbf{y}$ satisfy (3.59) separately. We thus know that inserting the right initial conditions will give us a real solution, or alternatively can just take the real part of whatever solution we find. Take the viewpoint that you find more comfortable.

Taking the latter view point (ii), we can also get rid of complex numbers at this point: Let us write the real part of each term in curved brackets in (3.69):

$$\begin{aligned} 2\mathrm{Re}\left(c_\ell^+ e^{i\omega_\ell t} + c_\ell^- e^{-i\omega_\ell t}\right) &= c_\ell^+ e^{i\omega_\ell t} + c_\ell^- e^{-i\omega_\ell t} + c_\ell^{+*} e^{-i\omega_\ell t} + c_\ell^{-*} e^{i\omega_\ell t} \\ &= \underbrace{\left(c_\ell^+ + c_\ell^{-*}\right)}_{\equiv A_\ell e^{i\varphi_\ell}/2} e^{i\omega_\ell t} + \underbrace{\left(c_\ell^- + c_\ell^{+*}\right)}_{\equiv A_\ell e^{-i\varphi_\ell}/2} e^{-i\omega_\ell t} \\ &= A_\ell \left(e^{i\omega_\ell t} e^{i\varphi_\ell} + e^{-i\omega_\ell t} e^{-i\varphi_\ell}\right)/2 = A_\ell \cos(\omega_\ell t + \varphi_\ell). \end{aligned} \tag{3.70}$$

In the second line, we wrote the pre-factors of complex exponentials in polar notation, using real numbers $A_\ell = 2|c_\ell^+ + c_\ell^{-*}|$ and $\varphi_\ell = \arg[c_\ell^+ + c_\ell^{-*}]$ and in the last line we used Eq. (3.68).

Alltogether we can now write the real solution as follows:

Normal modes and normal frequencies for two coupled oscillators

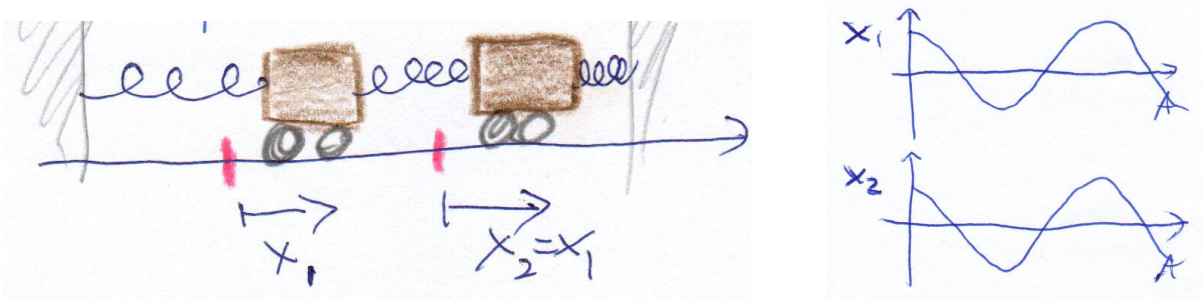
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_0 t + \varphi_0) + A_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_1 t + \varphi_1). \tag{3.71}$$

where the A_k are normal mode amplitudes, the ω_k are normal mode frequencies $\omega_0 = \sqrt{k/m}$, $\omega_1 = \sqrt{3k/m}$, the φ_k are normal mode phases and the vectors $[1,1]^T$ and $[1,-1]^T$ describe the normal modes themselves.

- We still have unknown numbers A_k, φ_k in the solution that need to be determined by the initial conditions.
- If we preferred, we could have written sin instead of cos, since that just changes φ_k.

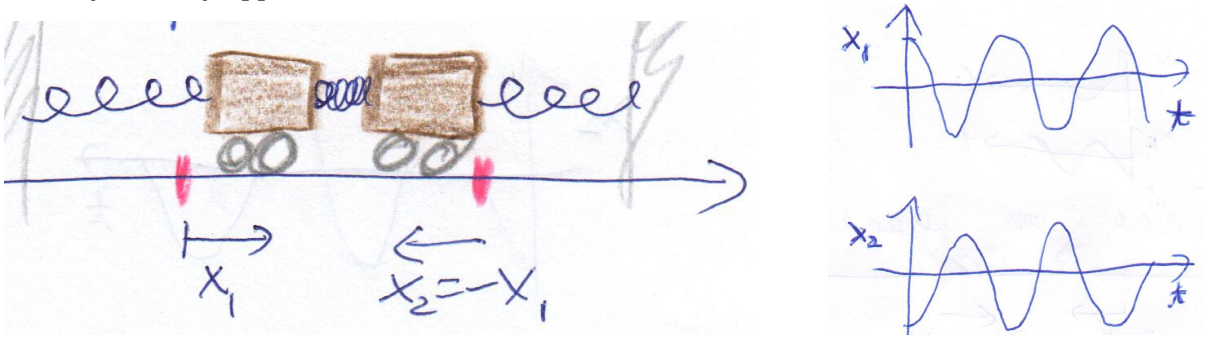
Let us now look at what these normal modes mean. Firstly we see from Eq. (3.71), that whenever only one A_k is non-zero, the motion will involve simple harmonic oscillation (but pertaining to both carts at once).

We have a symmetric normal mode for $A_1 = 0$, then $x_1(t) = x_2(t) = A_0 \cos(\omega_0 t + \varphi_0)$, thus both carts move perfectly synchronized. This motion is visualized below, for zero phase offset $\varphi_0 = 0$.



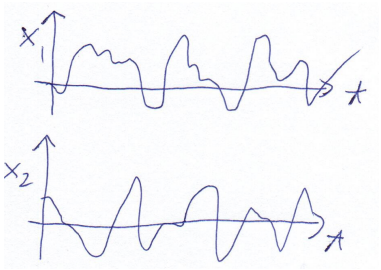
We see that in this motion the middle spring is actually never stretched, so it makes sense that the carts move at the same frequency as if they were individually attached to the wall only.

The second normal mode is anti-symmetric: $x_1(t) = -x_2(t) = A_1 \cos(\omega_1 t + \varphi_1)$, so the cart motion is always exactly opposite:



In this mode the middle spring is heavily involved, so the normal mode frequency differs from the single oscillator one.

The differential equation (3.59) is linear in $\mathbf{x}$, so it obeys the superposition principle: Any linear combination of valid solutions will itself be a solution. We can thus express the most general solution (3.71) as a superposition of the two normal modes. Depending on the initial conditions, usually both modes would be involved, which can look rather irregular:



To see everything we just learnt in an experiment, watch this video .

Weakly coupled oscillators:

A second tractable case is if the coupling between the oscillating carts is weak, so for the second spring $k_2 \ll k_1,\ k_3$ for $k_1 = k_3 = k$. We still keep $m_1 = m_2 = m$. We can proceed as before, but instead of (3.66) we now get

$$\tilde{M} = \begin{bmatrix} \frac{k+k_2}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k+k_2}{m} \end{bmatrix}, \tag{3.72}$$

The eigenvectors of this matrix are actually the same as of (3.66), but one of the eigenvalues changes, so we now have eigenmode frequencies

$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad \omega_1 = \sqrt{\frac{k+2k_2}{m}}. \tag{3.73}$$

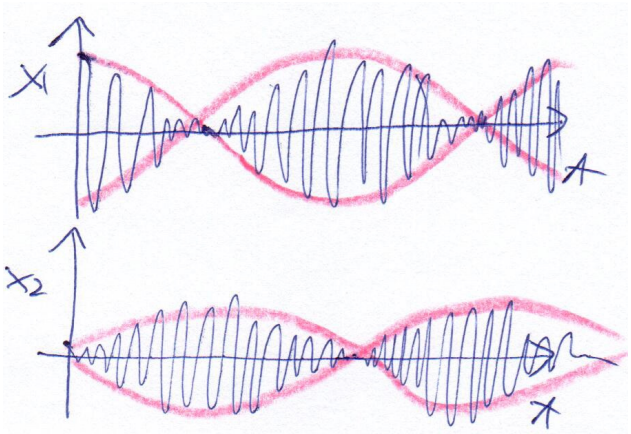
Since $k_2 \ll k$, these two frequencies are almost the same. Let us define $\bar{\omega} = (\omega_0 + \omega_1)/2$ and $\epsilon = (\omega_1 - \omega_0)/2$, then we know that $\bar{\omega} \approx \omega_0 \approx \omega_1$ and $\epsilon \ll \bar{\omega}$ is small.

Inspection of individual normal modes would give very similar result as in the three equal springs case, but for the weakly coupled case we can make sense of the combination of normal modes.

Let us insert the frequencies into (3.71), set $A_0 = A_1 = A/2$ and $\varphi_0 = \varphi_1 = 0$ and return to the complex notation by replacing $\cos(\omega t)$ with $\exp(i\omega t)$. We then have

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \mathrm{Re}\left\{ \frac{A}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0-\epsilon)t} + \frac{A}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0+\epsilon)t} \right\} \\ &\overset{Eq.\ (3.68)}{=} \mathrm{Re}\left\{ A \begin{bmatrix} \cos(\epsilon t) \\ -i\sin(\epsilon t) \end{bmatrix} e^{i\omega_0 t} \right\} = \begin{bmatrix} \cos(\epsilon t)\cos(\omega_0 t) \\ \sin(\epsilon t)\sin(\omega_0 t) \end{bmatrix} \end{aligned} \tag{3.74}$$

Since we know that $\epsilon \ll \omega_0$, these look as shown below. Each $x_k(t)$ rapidly oscillates with the mean frequency ω_0. However the amplitude of these oscillations migrates between the two oscillators periodically, giving rise to a beat pattern.



left: Beating of two coupled harmonic oscillators. Fast oscillations have a period $T_{\text{fast}} = 2\pi/\bar{\omega}$ while the envelope changes on a much larger period $T_{\text{slow}} = 2\pi/\epsilon$ (for two beatings).

The same pattern is seen, when e.g. superimposing two sound or light waves of nearby frequency or wavelength.

3.6.2 Many coupled oscillators

So far we studied seemingly special scenarios. We will now show that these are actually highly representative of what is the most frequently encountered general dynamics. Consider a generic Lagrangian with M generalized coordinates q_n, which we group into a single M dimensional vector $\mathbf{q} = [q_1, \cdots q_M]^T$. We assume the system is conservative so that we can write a potential energy $V(\mathbf{q})$ which is now also a function in M-dimensions. Let the constraints be holonomic $\mathbf{r} = \mathbf{r}(\mathbf{q})$. We can then show that the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}\sum_{k\ell} A_{k\ell}(\mathbf{q})\dot{q}_k\dot{q}_\ell - V(\mathbf{q}). \tag{3.75}$$

For this, you can show that e.g. $A_{k\ell}(\mathbf{q}) = \sum_\alpha m_\alpha(\partial\mathbf{r}_\alpha/\partial q_k)(\partial\mathbf{r}_\alpha/\partial q_\ell)$, assuming that the unconstrained kinetic energy was $T = \frac{1}{2}\sum_\alpha m_\alpha \mathbf{r}_\alpha^2$, however the precise form of $A_{k\ell}(\mathbf{q})$ is not so important in the following. You can look at assignment4 and quiz1 for cases where the function $A_{k\ell}(\mathbf{q})$ was non-trivial and the kinetic energy contained cross-terms ($\sim \dot{q}_a\dot{q}_b$).

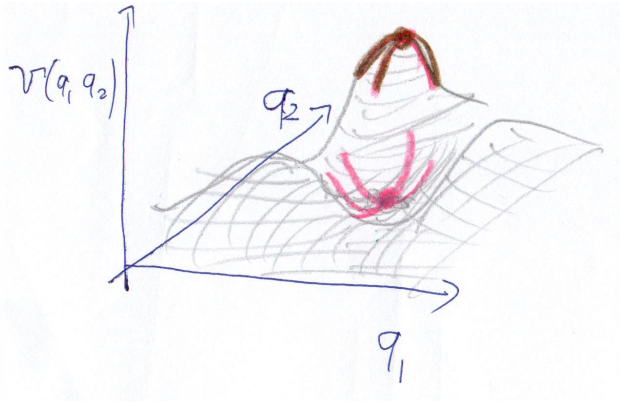
We can see from Eq. (2.29), that a special point in coordinate space is given by

Equilibrium points We say a mechanical system is in equilibrium at a point $\mathbf{q}_0$ for which all generalized forces vanish:

$$\frac{\partial\mathcal{L}}{\partial q_n} = -\frac{\partial V(\mathbf{q})}{\partial q_n} = 0. \tag{3.76}$$

Hence all generalized momenta $\dot{q}_n$ remain zero if initially zero.
The equilibrium is called stable, if a small perturbation $\delta\mathbf{q}(t)$ around this point $\mathbf{q}(t) = \mathbf{q}_0 + \delta\mathbf{q}(t)$ will remain small (e.g. oscillate) and unstable of that is not the case.

We see in the figure below, how this looks in some arbitrary high dimensional energy landscape:



left: Sketch of some potential energy with two generalized coordinates q_1, q_2. All extremal points of this function satisfy $\partial V(\mathbf{q})/\partial q_n = 0$ and are hence equilibrium points, this includes maxima, minima and saddle points. We have drawn the cuts through this surface along the coordinate axes in brown (near a maximum) and pink (near a minimum)

- Of the extrema, only local minimal will be stable, saddle points and maxima are unstable.
- We had seen some first examples of stable/unstable dynamics in assignment 3 Q4 (the hoop on a ring), or assignment 5 Q2, the stability of rotation around the different principal axes.
- I cannot draw more than two dimensions, but you can imagine that all the same concepts apply in any number of dimensions/ for any number of coordinates.

It is more likely to find real physical systems in the vicinity of a stable equilibrium (because otherwise they would break or be driven from an unstable to a stable point, in the presence of dissipation). Since the system will typically not make more than small excursions away from $\mathbf{q}_0$, we can use a multi-variate Taylor expansion of the potential energy around the equilibrium point

$$V(\mathbf{q}) \approx V(\mathbf{q}_0) + \sum_k \underbrace{\left.\frac{\partial V(\mathbf{q})}{\partial q_k}\right|_{\mathbf{q}=\mathbf{q}_0}}_{=0} (q_k - q_{0k}) + \frac{1}{2}\sum_{k\ell} \underbrace{\left.\frac{\partial^2 V(\mathbf{q})}{\partial q_k \partial q_\ell}\right|_{\mathbf{q}=\mathbf{q}_0}}_{\equiv K_{k\ell}} (q_k - q_{0k})(q_l - q_{0\ell}) \tag{3.77}$$

The linear term of the Taylor expansion is zero, due to our definition (3.76) of an equilibrium point. For the remainder, let us assume we can change coordinates such that $\mathbf{q}_0 = 0$ and adjust our zero of energy such that $V(\mathbf{q}_0) = V(0) = 0$. Finally we give the coefficients of the second order expansion terms the name "matrix K" as shown, and are done with simplifying the potential energy.

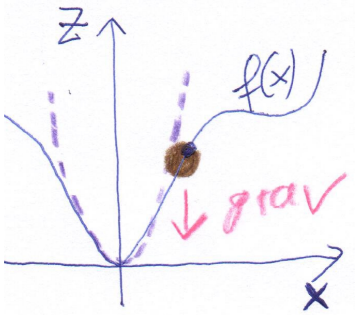
For the kinetic energy, we want to expand everything to second order in $\dot{q}_k$. Since the term $\dot{q}_k\dot{q}_\ell$ in (3.75) already is of second order, we only need the constant part $M_{k\ell} = A_{k\ell}(0)$ of the prefactor. With that, we finally arrive at a much simplified

Lagrangian near an equilibrium point

$$\mathcal{L} = \frac{1}{2}\sum_{k\ell} \left(M_{k\ell}\dot{q}_k\dot{q}_\ell - K_{k\ell}q_k q_\ell\right). \tag{3.78}$$

- This is called a quadratic form. In vector notation you can write it as $\mathcal{L} = \frac{1}{2}\left(\dot{\mathbf{q}}^T M\dot{\mathbf{q}} - \mathbf{q}^T K\mathbf{q}\right)$.
- We know that M and K are symmetric matrices, and shall use that shortly.

Example 38, Bead on a wire: Consider the bead of mass m constrained to sit on an arbitrarily shaped wire shown in the figure below.



left: Bead (brown), stuck on a wire defined by coordinates $z = f(x)$, subject to gravity $V_{\text{grav}} = mgz$ (pink).

We want to use a generalized coordinate $q = x$. As we had seen in quiz one, starting from kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2)$ and then using the constraint $z = f(x)$, we find $T = \frac{1}{2}m\left[1 + \left(\frac{\partial f(q)}{\partial q}\right)^2\right]\dot{q}^2$ for the kinetic part of the Lagrangian and $V = mgf(q)$ for the potential one. When expanding to second order in $\dot{q}$ and q we can drop the second piece of that kinetic energy, and Taylor expand the potential around the minimum (violet dashed line in figure).

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\underbrace{g\frac{\partial^2 f(q)}{\partial q^2}}_{=\omega^2}q^2 \tag{3.79}$$

We see directly that despite the more complicated starting point, we reached the Lagrangian for a simple harmonic oscillator.

We now immediately proceed to find Lagrange's equations from (3.78). For this we can use the Kronecker delta symbol δ_{nm} introduced near (2.98) to write e.g. $\partial q_n/\partial q_m = \delta_{nm}$. Then

$$\frac{\partial \mathcal{L}}{\partial q_n} = -\frac{1}{2}\sum_{k\ell} K_{k\ell}\frac{\partial}{\partial q_n}(q_k q_\ell) = -\frac{1}{2}\sum_{k\ell} K_{k\ell}(\delta_{nk}q_\ell + q_k\delta_{n\ell}) = -\frac{1}{2}\left(\sum_\ell K_{n\ell}q_\ell + \sum_k K_{kn}q_k\right) = -\sum_\ell K_{n\ell}q_\ell. \tag{3.80}$$

where in the last equality we have used that the matrix K is symmetric ($K_{k\ell} = K_{\ell k}$). In the same way, we find

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_n} = \sum_\ell M_{n\ell}\dot{q}_\ell, \tag{3.81}$$

hence we reach the

Lagrange equations for dynamics near an equilibrium point

$$M\ddot{\mathbf{q}} = K\mathbf{q} \quad \Leftrightarrow \quad \ddot{\mathbf{q}} = M^{-1}K\mathbf{q} \tag{3.82}$$

- We have used matrix-vector product notation again.

- For the equivalence written, we of course require that the matrix M is invertible. In the second form, you can then see, that Eq. (3.59) was a special case of this form.
- Our general solution method (3.65) applies straightforwardly to Eq. (3.82).

Similar to Eq. (3.71) we find the oscillator solutions of Eq. (3.82) as:

Normal modes and normal frequencies for oscillations near equilibrium

$$\mathbf{q}(t) = \mathrm{Re}[\sum_\ell a_\ell \mathbf{v}_\ell e^{i\omega_\ell t}]. \tag{3.83}$$

As before ω_ℓ are called normal mode frequencies, $\mathbf{v}_\ell$ normal modes, and their respective coefficients a_ℓ are fixed by the initial conditions. The ω_ℓ are found as $\omega_\ell = \sqrt{\lambda_\ell}$ from the non-zero eigenvalues of $M^{-1}K$, the normal modes are the corresponding eigenvectors.

- Since we started with a very general mechanical problem near equilibrium at the beginning of section 3.6.2, we now see that a large portion of physics can be understood in terms of (coupled) harmonic oscillators.
- If you prefer, you can do similar steps to reduce Eq. (3.83) to cosines with arbitrary phase, as we did for Eq. (3.71). However it actually typically is easier to work with complex exponentials as far as possible.
- After finding eigenvalues and eigenvectors of $M^{-1}K$ we can again use a transformation matrix O as in (3.29) to define new coordinates $\tilde{\mathbf{q}} = O\mathbf{q}$ that bring the Lagrange equation into diagonal form:

  $$\ddot{\tilde{q}}_\ell = -\omega_\ell^2 \tilde{q}_\ell. \tag{3.84}$$

  Since these now describe a system of uncoupled harmonic oscillators, the functioning of the normal modes becomes more apparent.

Two alternative solution methods for second order system:
Method 1: We can directly make the Ansatz[a] $\mathbf{q}(t)_\ell = a_\ell e^{i\omega_\ell t}$. Insertion into (3.82) gives

$$K\mathbf{a} = \omega^2 M\mathbf{a} \quad \Leftrightarrow \quad (K - \omega^2 M)\mathbf{a} = 0 \tag{3.85}$$

This is called a generalized eigenvalue problem. We say $\mathbf{a}$ is the eigenvector of K with respect to M. The solution proceeds similar to the usual eigenvalue problem (and hence we can also get away without knowing the name above). Using the expression to the right of $\Leftrightarrow$ above, we note that to have a non-zero solution for $\mathbf{a}$ we require $\det(K - \omega^2 M) = 0$. The resultant polynomial equation gives us the eigenvalues ω^2 in the usual way. Subsequently we can add up all solutions $\mathbf{q}(t)_\ell$ that we found, since Eq. (3.82) is linear and obeys the superposition principle. We then reach Eq. (3.83) again.

[a] As discussed before, we can work with the complex Ansatz and take the real part in the end.

continued: *Method 2:* We can always reduce a system of second order differential equations to a (larger) system of first order differential equations with the following trick: Define $\mathbf{u}(t) = \dot{\mathbf{q}}(t)$, and combine $\mathbf{y}(t) = [\mathbf{q}, \mathbf{u}]^T$ into a large $2M$ dimensional vector. We can then write Eq. (3.82) as

$$\dot{\mathbf{y}}(t) = \begin{bmatrix} 0 & \mathbb{1} \\ M^{-1}K & 0 \end{bmatrix} \mathbf{y}(t), \tag{3.86}$$

where all the four items in the matrix above are themselvs $M \times M$ matrix blocks. At this point we can use solution methods for first order systems, such as Eq. (3.63). See also the numerics part of assignments 1 or 4.

We shall apply the complete formalism to examples in assignments and tutorials.