Week 8

PHY 305 Classical Mechanics

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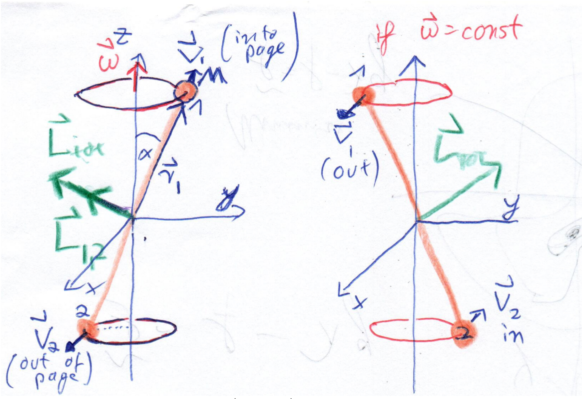
In week7 we discussed some kinematics of rotational motion of rigid bodies, mainly how angular momentum and rotational velocity are linked. We did not yet get to the dynamics of rotational motion, i.e. how angular momentum and rotational velocity evolve in time, given an initial state. It turns out the dynamics of rotating objects can be amazingly counterintuitive, see this video or this one , if you prefer some talking. In this week, will try to understand the effect shown in those videos based on our results from week7. We will explain these videos in section 3.5.2.

Before, let's revisit another confusing aspect of rotational motion: We already had hinted at in example 28 that a scenario where the rotation axis is not parallel to the angular momentum will lead to interesting complications once we move to time evolution / dynamics. Let us revisit this in a slightly simpler example:

Example 32, Tumbling of tilted dumbbell:



left: If in example 28 we consider the entire mass restricted to the endpoints of the stick, we reduce the problem to two mass points at shown, rotating about the z-axis at time $t = 0$ (left). Assume both mass points are in the yz plane (at $x = 0$) with rotational velocity at shown, but this time not constrained by any mounting.

You can use Eq. (3.12) and the right hand rule to convince yourself that mass M_1 moves into the paper and M_2 out of it. Then you can use the right hand rule again, to confirm that the angular momentum $\mathbf{L}$ for both masses points in the same direction (green) also in the yz plane.

If the masses were now just rotating with a constant rotational velocity $\boldsymbol{\omega}$, some time later they would be at the locations shown on the right, with velocities reversed compared to before. Another application of the right hand rule give the angular momentum (green) now pointing in a different direction. However, we know from section 1.4.5 and/or section 2.7.2 that for an isolated system, angular momentum is conserved.

The only way to salvage this contradiction, is to allow that the rotation axis $\boldsymbol{\omega}$ does in fact change in time $\boldsymbol{\omega}(t)$.

3.2 Free precession

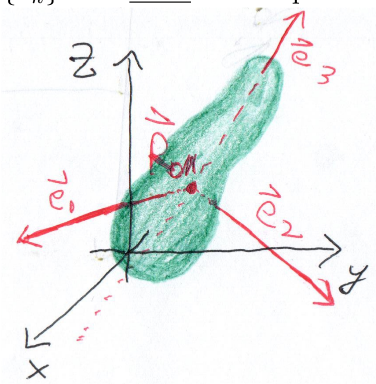
After considering the above example, we should of course investigate how the rotation axis will evolve in general. In this section we first consider the case without any external torque, e.g. $\mathbf{N}^{(e)} = 0$ in Eq. (1.20). We then combine Eq. (1.20) and Eq. (3.26) to reach in a space-fixed frame (lab frame):

$$0 = \dot{\mathbf{L}} = \frac{d}{dt}\big(I\boldsymbol{\omega}(t)\big). \tag{3.33}$$

Angular momentum is conserved, see also section 2.7.2, however since all components of the (3×3) matrix I are nonzero, it is difficult yo use Eq. (3.33) in this form to infer $\boldsymbol{\omega}(t)$. It would be easier to use the diagonalized form (3.29) of I with the principal axes as coordinates, which we explore in the next section.

3.2.1 Space versus body frames

We had mentioned below Eq. (3.28) that there are always three real and orthogonal 3-component eigenvectors of the inertia tensor for any rigid body. This means we can choose these eigenvectors $\{\mathbf{e}_k\}$ as a basis of 3D-space as shown in the figure below.



left: (left) A rigid object can be either described in a usual space fixed coordinate frame (black) or alternatively in a body-fixed, rotating frame where coordinate axes are chosen as the principal axes of the body (red).

We can thus chose to either work in a space-fixed frame, our usual frame with basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ or a body-fixed frame, with basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. To convert any vector written down in the space fixed frame $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ to one in the body fixed frame $\mathbf{r}' = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3$, we multiply it with the transformation matrix O defined in Eq. (3.29).

$$\mathbf{r}' = O^T\mathbf{r}, \quad \mathbf{r} = O\mathbf{r}'. \tag{3.34}$$

We shall write vectors and vector components in the body frame with a prime as in Eq. (3.34). The advantage to work in the body-fixed frame, is that the inertia tensor I is diagonal (per definition). The disadvantage is, that the body-fixed frame is, in general, a rotating frame and thus non-inertial. However we already learnt how to deal with non-inertial frames in section 2.9 and thus can use those results.

3.2.2 Euler's equations

Most importantly, we can use Eq. (2.87) to see that

$$0 \overset{Eq.\ (3.33)}{=} \left(\frac{\partial \mathbf{L}}{\partial t}\right)_{\text{space}} = \left(\frac{\partial \mathbf{L}}{\partial t}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L}. \tag{3.35}$$

We now insert (3.26) for $\mathbf{L}$ in the body fixed frame, where it gives $L_k = \lambda_k \omega_k$, for $k = 1, 2, 3$, and then resolve Eq. (3.35) into vector components using $\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$ to reach

Euler's equations for the case of no external torque:

$$\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2\omega_3, \\ \lambda_2 \dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3\omega_1, \\ \lambda_3 \dot{\omega}_3 &= (\lambda_1 - \lambda_2)\omega_1\omega_2, \end{aligned} \tag{3.36}$$

govern the evolution of the rotation axis $\boldsymbol{\omega}(t)$ of a rigid body.

- Eq. (3.36) now tell us how the rotation axis will evolve in the body-fixed frame.
- ω_k with $k \in \{1, 2, 3\}$ are the components of the rotational velocity of the rigid body when expressed in the body-fixed frame, but they do not imply a "rotation in the body-fixed frame" (which would vanish per definition).
- There is a more complete version of Euler's equation with external torque, however in that case it becomes less useful since we ought to express the external torque in the body fixed frame, making that cumbersome.
- We see that if at $t = 0$, the rotational velocity $\boldsymbol{\omega}$ points along a principal axis, it will be constant ($\dot{\boldsymbol{\omega}} = 0$). This is because in that case two out of three ω_k are zero, so all right hand sides in (3.36) must be zero.
- Also conversely, if the rotation axis is initially not pointing along a principal axis, at least two components ω_k must be nonzero, then at least one right had side of (3.36) is nonzero, so at least one $\dot{\omega}_k$ is nonzero, hence the rotation axis will change in time.
- You can use Euler's equation to show that rotation is stable when the rigid body is rotating around a principal axis that has either the largest or smallest moment of inertia, and unstable otherwise (see assignment 5). Stable means that if you perturb the axis slightly, the axis will just undergo small oscillations near the principal axis, instead of being driven away entirely.

3.2.3 Precession of rotation axis

We can solve Eq. (3.36) completely for a rigid body that has two principal axes with equal moments of inertia, let us assume $\lambda_1 = \lambda_2$. In that case we immediately see $\omega_3 = const$ from the third equation of (3.36). In that case we can rewrite the first two equations as

$$\dot{\omega}_1 = \underbrace{\frac{(\lambda_1 - \lambda_3)\omega_3}{\lambda_1}}_{\equiv \Omega_b} \omega_2 = \Omega_b \omega_2,$$

$$\dot{\omega}_2 = -\frac{(\lambda_1 - \lambda_3)\omega_3}{\lambda_1}\omega_1 = -\Omega_b \omega_1. \tag{3.37}$$

Fro this we have defined the constant frequency Ω_b. We can solve Eq. (3.37) by differentiating the first one in time and then inserting the second (or the reverse), to reach $\ddot{\omega}_{1,2} = -\Omega_b^2 \omega_{1,2}$, with the solution:

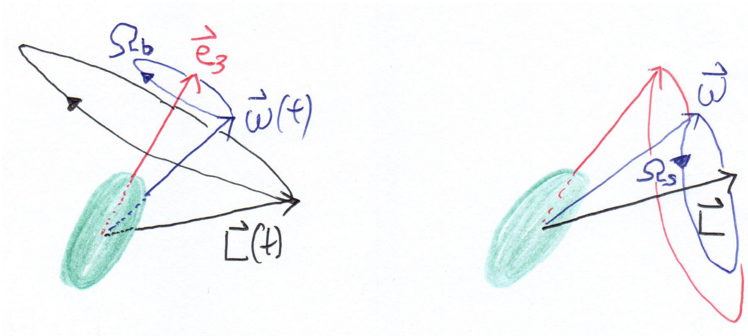
Rotational velocity precessing on the body cone
In the body fixed frame, we find

$$\boldsymbol{\omega} = [\omega_0 \cos(\Omega_b t), -\omega_0 \sin(\Omega_b t), \omega_3]^T \tag{3.38}$$

and hence using (3.26) ($L_k = \lambda_k \omega_k$) also

$$\mathbf{L} = [\lambda_1 \omega_0 \cos(\Omega_b t), -\lambda_1 \omega_0 \sin(\Omega_b t), \lambda_3 \omega_3]^T \tag{3.39}$$

The motion of both vectors is depicted in the figure below, in the body-frame as for Eq. (3.38)-Eq. (3.39) and in the space frame.



top: Principal axis (red), rotation axis (blue) and angular momentum (black) in the body-fixed frame (left) and in the space frame (right). The precession of the vectors in time, in either frame, is indicated by circles.

- Let's remain in the body frame first. From Eq. (3.38) you can see that $\boldsymbol{\omega}$ is precessing such that the vector traces out a cone around $\mathbf{e}_3$, with precession frequency

$$\Omega_b = \frac{(\lambda_1 - \lambda_3)\omega_3}{\lambda_1}. \tag{3.40}$$

  Thus after $\tau_b = 2\pi/\Omega_b$, the rotation axis has completed one period of precession in the body-fixed frame.

- By taking the scalar product $\boldsymbol{\omega}(t) \cdot \mathbf{L}(t)$ using Eq. (3.38)-Eq. (3.39), you can convince yourself that the 3D angle between $\boldsymbol{\omega}(t)$ and $\mathbf{L}(t)$ is constant. Thus all three vectors must remain in one plane, so the angular momentum is precessing around $\mathbf{e}_3$ with the same frequency as $\boldsymbol{\omega}(t)$ but a different angle.

- When going from the body-frame to the space frame, we know that it is $\mathbf{L}$, which must be constant, due to angular momentum conservation.
- The conversion from the body-frame to the space-frame is just a rotation, with time-dependent rotation axis. Rotations do not change (i) angles, and (ii) the fact that three vectors lie in one plane, so both of these are true in the space-frame as well.
- It is then hopefully reasonable, that we reach the picture on the rhs above, where $\mathbf{L}$ is constant and the plane containing $\mathbf{e}_3$ and $\boldsymbol{\omega}$ rotates around that. In this frame, we will find a different precession frequency $\Omega_s = L/\lambda_1$. *Note: Showing all these statements mathematically rigorously is technically tricky. Please see Morin [MM] (newly added book-reference, week 0) for some more details.*

You can attempt to test many of the statements above in your room by experiments, throwing objects up such that they are rotating and trying to observe the evolution of their rotation axes. While they are falling up and down, the fictitious force (2.80) in their rest frame cancels gravity ("free fall"), such that there is no external torque. However the need to be in free fall, restricts the duration of your experiment. More long term realisation of a (nearly) torque free rotation are provided in our solar system, for example:

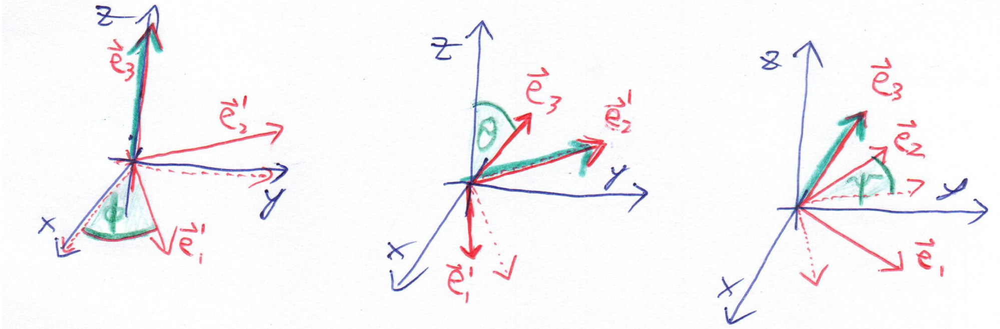
Example 33, The asteroid Toutatis: Most asteroids actually rotate around principal axes (would you expect that?, why is that?), however a few don't. One example is called "Toutatis" and has passed sufficiently close to earth on one instance, that we could make detailed radar map recordings of its rotation as seen in this video and the file `toutspin.mpg` online.

Note that the asteroid has no perfect symmetry axes but, like all rigid objects, has three well defined principal axes anyway. It has three different moments of inertia, so the rotational axis dynamics is more complex than in the discussion of this section.

After having dealt with the torque free case, let us now move to rigid body rotation in the presence of an external torque. We want to use the Lagrangian formalism for that. However from the discussion after Eq. (2.29), we know that while we can use the Lagrange equations also in a non-inertial frame (such as the body-fixed frame), we have to initially set up the Lagrangian in an inertial frame (such as the space frame). We thus require more formal concepts for conversion between the two frames, which we set up nextly.

3.3 Euler Angles

As stated in section 3.1.1, there are three degrees of freedom associated with rotation. It should thus be possible to decompose a general rotation into three separate rotations about some special axes. One specific procedure/convention to do this decomposition is the use of Euler angles. Be careful, there are different definitions for what is meant by "Euler angles" and then there are different procedures alltogether. The use of Euler angles is sketched in the figure below:



top: Euler's angles quantify a sequence of three rotations about pre-defined axes, in order to map xyz (the space fixed frame) onto $e_1e_2e_3$ (the body-fixed frame). The blue (space) coordinate system never changes. We draw red before (dashed) and after (solid) rotation in each step. Each of the three rotations is about the cyan axis by the green angle.

We follow the sequence

(i) We rotate by an angle ϕ around the axis $\hat{\mathbf{z}}$. This angle ϕ is the final desired azimuthal angle of the body principal axis $\mathbf{e}_3$. This rotation turns the initial $\mathbf{e}_2$ into $\mathbf{e}_2'$ (we have to give it a name, since we need it in the next step).

(ii) We rotate by an angle θ around the axis $\mathbf{e}_2'$. This angle θ is the final desired polar angle of the body principal axis $\mathbf{e}_3$. $\mathbf{e}_3$ is thus now pointing in its final direction.

(iii) We thus cannot change the direction of $\mathbf{e}_3$ any more, and instead do a rotation around $\mathbf{e}_3$ by an angle ψ, such that $\mathbf{e}_1$ and $\mathbf{e}_2$ reach their final positions. Since all three vectors $\mathbf{e}_k$ are mutually orthogonal, this must be possible.

To set up a Lagrangian, we need to expression rotational velocities and angular momenta of a rigid body in the space fixed frame, taking as dynamical quantities Euler angles $\phi(t)$, $\theta(t)$, $\psi(t)$ that rotate the rigid body as in the sequence above. If we consider the first step only, and compare back to the diagrams in section 2.9.2, then as ϕ varies in time the first Euler rotation is a rotation with angular velocity $\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{z}}$. We know from (2.85) that rotational velocities add up, so the final rotational velocity of the the three steps in the figure above will be

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\mathbf{e}_2' + \dot{\psi}\mathbf{e}_3. \tag{3.41}$$

This has the disadvantage that it mixes unit vectors from different frames, so we use $\hat{\mathbf{z}} = \cos\theta\mathbf{e}_3 - \sin\theta\mathbf{e}_1'$ (exercise, see figure above second panel). to write

$$\boldsymbol{\omega} = \underbrace{(-\dot{\phi}\sin\theta)}_{=\omega_{1'}}\mathbf{e}_1' + \dot{\theta}\mathbf{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3. \tag{3.42}$$

Later we will only cover the rotation of objects called a symmetric top, which have two equal moments of inertia (as in section 3.2.3). In that case the body axes 1 and 2 can anyway be arbitrarily rotated, hence we do not need to do the final step to replace $\mathbf{e}_{1,2}'$ by $\mathbf{e}_{1,2}$. From Eq. (3.42) we later also need the expression for the angular momentum in terms of Euler angles, which is easy in the body-fixed frame. Using $L_k = \lambda_k \omega_k$, we find

$$\mathbf{L} = (-\lambda_1 \dot{\phi} \sin\theta)\mathbf{e}_1' + \lambda_1 \dot{\theta}\mathbf{e}_2' + \underbrace{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}_{=L_3}\mathbf{e}_3. \tag{3.43}$$

We now want to write down the kinetic energy in the inertial space frame for later use in Lagrange's equation. From Eq. (3.32) we know that it is $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$. Now we use the trick that a scalar product takes the same value in any coordinate system and evaluate it in the body frame, which has the advantage that we can use $L_k = \lambda_k \omega_k$. Taking the scalar product of (3.42) and (3.43), we reach

$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2. \tag{3.44}$$

One final result that we need later, and that, with a few technical steps you can derive from $L_k = \lambda_k \omega_k$ and converting that back to find L_z is

$$\dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}. \tag{3.45}$$

3.4 Rotation matrices [BONUS MATERIAL]

Understanding the relations above, involved quite a lot of technical thinking about axes etc. It would be nice to have a more automatic method to handle rotation, this is provided by rotation matrices O, which we already encountered around Eq. (3.29). Let us define more formally a

Rotation matrix, as an orthogonal matrix O , which when multiplied to any vector $\mathbf{r}$ expressed in a coordinate system with basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ gives the representation of that vector $\mathbf{r}'$ in a rotated coordinate system with basis vectors $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, $\hat{\mathbf{z}}'$

$$\mathbf{r}' = O^T\mathbf{r}. \tag{3.46}$$

Using the Euler angles introduced above, we can write:

$$O^T = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\psi s_\phi - c_\phi c_\theta s_\psi & c_\phi s_\theta \\ c_\phi s_\psi + c_\theta c_\psi s_\phi & c_\phi c_\psi - c_\theta s_\phi s_\psi & s_\phi s_\theta \\ -c_\psi s_\theta & s_\theta s_\psi & c_\theta \end{bmatrix}. \tag{3.47}$$

where $c_\alpha = \cos\alpha$ and $s_\alpha = \sin\alpha$.

- Most of the calculations of week 6,7,8 could have been expressed using rotation matrices instead of our approach, however that would not necessarily make it simpler.
- However rotation matrices play a major role in considering symmetries involving rotations in quantum physics, using group theory. For that reason, it is useful to know that these matrices form a group (see math), which e.g. for $N \times N$ matrices is called $SO(N)$ ($O(N)$ would be the group of orthogonal $N \times N$ matrices, and the "S" specifies $\det O = 1$).

3.5 Dynamics of a spinning top

In this section, we will finally discuss the dynamical precession of a spinning top, which was the subject of the movies linked at the beginning of "week 8". The main difference between that problem and section 3.2, is that now we have to consider the torque exerted by gravity.

3.5.1 Net torque

We had seen in section 1.4.5/Eq. (1.20), that for a collection of massive objects the total angular momentum changes according to $\dot{\mathbf{L}} = \mathbf{N}^{(e)}$ in response to the net external torque $\mathbf{N}^{(e)} = \sum_k \mathbf{r}_k \times \mathbf{F}_k^{(e)}$. For completeness, let us give the torque also in terms of the continuum notation introduced for rigid bodies in section 3:

External torque acting on a rigid body with respect to the chosen origin is

$$\mathbf{N}^{(e)} = \sum_k \mathbf{r}_k \times \mathbf{F}_k^{(e)} = \int d^3\mathbf{r}\ \mathbf{r} \times f(\mathbf{r}), \tag{3.48}$$

where $f(\mathbf{r})$ is the force density acting on location $\mathbf{r}$.

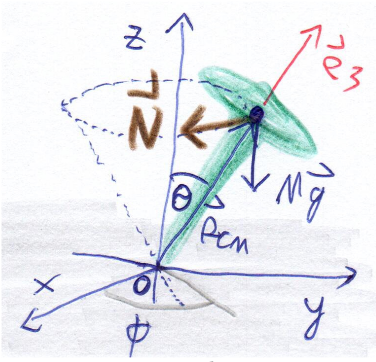
- The case of particular relevance shortly, is a torque exerted by gravity. Then $f(\mathbf{r}) = -\rho(\mathbf{r})g\hat{\mathbf{z}}$ and hence

$$\begin{aligned}\mathbf{N}^{(e)} &= -g\int d^3\mathbf{r}\,\rho(\mathbf{r})\,\mathbf{r}\times\hat{\mathbf{z}} = -g\left(\int d^3\mathbf{r}\,\rho(\mathbf{r})\,\mathbf{r}\right)\times\hat{\mathbf{z}} = -gM\frac{\left(\int d^3\mathbf{r}\,\rho(\mathbf{r})\,\mathbf{r}\right)}{M}\times\hat{\mathbf{z}}\\ &= -gM\mathbf{R}_{CM}\times\hat{\mathbf{z}} = \mathbf{R}_{CM}\times\underbrace{\mathbf{F}_{\text{grav,M}}}_{\equiv -gM\hat{\mathbf{z}}}.\end{aligned} \tag{3.49}$$

  We thus see that the net external torque due to gravity can be found simply by assuming all mass is concentrated at the centre of mass.

3.5.2 Precession in a simplified picture

We first tackle the precession of a top in a simple Newtonian approach, to get a hang of things. The geometry is illustrated in the figure below.



left: Spinning top (green) with figure symmetry axis/ principal axes (red). We assume the point of contact with the table (grey) remains constant at the origin O. Gravity acts downwards, but that gives rise to a sideways torque (brown) [This should be pointing into the paper, so is in the wrong direction in the diagram]. If this torque is weak, it gives rise the rotation axis precessing on a cone (blue dashed).

Let us assume the top is initially rotating rapidly about its symmetry (and principal) axis $\mathbf{e}_3$ as shown. Then from $L_k = \lambda_k \omega_k$, we have $\mathbf{L} = \lambda_3 \boldsymbol{\omega} = \lambda_3 \omega_3 \mathbf{e}_3$. Since the only obvious fixed point of the top is the point of contact with the table, we chose that point as our origin O.

From the previous section 3.5.1, we know gravity will exert a torque $\mathbf{N} = M\mathbf{R}_{CM} \times \mathbf{g}$, with $\mathbf{g} = -g\hat{\mathbf{z}}$ on the top. Using the right-hand-rule and the diagram, you can see that the torque is a vector that is pointing into the paper, orthogonal to $\mathbf{R}_{CM}$. By construction here, the torque is thus initially orthogonal to $\mathbf{L}$, so based on $\dot{\mathbf{L}} = \mathbf{N}$ it can change the direction of angular momentum, but not its magnitude.

Let us make a major simplification, assuming that the angular momentum remains parallel to the rotation axis, and we can write $\mathbf{L}(t) = \lambda_3 \omega_3 \mathbf{e}_3(t)$ for constant λ_3, ω_3. Since we have seen in section 3.2 that the two vectors need not be parallel, this would have to be shown. It turns out to be a good approximation, for a weak torque.

We can then turn $\dot{\mathbf{L}} = \mathbf{N}$ into

$$\dot{\mathbf{L}} = \lambda_3 \omega_3 \dot{\mathbf{e}}_3(t) = M\mathbf{R}_{CM} \times \mathbf{g} = \mathbf{N}. \tag{3.50}$$

Since we know that $\mathbf{R}_{CM}(t) = R\mathbf{e}_3(t)$ from the diagram and $\mathbf{g} = -g\hat{\mathbf{z}}$, the middle equality becomes

$$\dot{\mathbf{e}}_3(t) = \frac{MgR}{\lambda_3 \omega_3} \hat{\mathbf{z}} \times \mathbf{e}_3(t) = \boldsymbol{\Omega} \times \mathbf{e}_3(t), \tag{3.51}$$

with precession axis

$$\boldsymbol{\Omega} = \frac{MgR}{\lambda_3 \omega_3} \hat{\mathbf{z}}. \tag{3.52}$$

By comparison of Eq. (3.51) with Eq. (2.84), we see that the vector $\mathbf{e}_3(t)$ will move on a cone around the precession axis that is parallel to $\hat{\mathbf{z}}$, as indicated in the figure, with an angular frequency $|\boldsymbol{\Omega}|$.

Example 34, Toppling top: A crucial input into the result above was that $\mathbf{L}$ is initially nonzero. If you start the top without rotation ($\mathbf{L}(t = 0) = 0$) at a tilted angle as in the figure? How will $\mathbf{L}$ evolve? Which motion does this describe?

Example 35, Precession in videos: You already saw it in the videos at the beginning of week 7. Here is another , which also explains the use of gyroscopes for aircraft navigation.

3.5.3 Precession in the Lagrange formalism

In general $\mathbf{L} \parallel \boldsymbol{\omega}$ need not be true. Hence we now re-solve the heavy symmetric[9] top problem in the Lagrange formalism. As usual we must start with the Lagrangian, but already did all the hard work in section 3.3 when deriving the kinetic energy of a rigid body in terms of Euler angles. We have to subtract the gravitational potential energy, which is however easy as it can be thought of just acting on the centre of mass (exercise). Hence we have the

Lagrangian of the heavy symmetric top as:

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta. \tag{3.53}$$

From this we derive the Lagrange equations (2.29) as usual and reach

Equations of motion of the heavy symmetric top

$$\lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2 \sin\theta\cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + MgR\sin\theta, \tag{3.54}$$

$$p_\phi \equiv \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \text{const} \tag{3.55}$$

$$p_\psi \equiv \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{const} \tag{3.56}$$

- It turns out the second equation (3.55) implies the conservation of the angular momentum projection onto the space fixed z-axis, L_z, even though that is not obvious to see.
- According to the third equation (3.56), also the component of the angular momentum along the figure axis, L_3, is conserved. You see this by comparison with Eq. (3.43).

We now want to re-derive the steady precession we had seen in section 3.5.2. For a steady precession, the angle θ of the body symmetry axis $\mathbf{e}_3$ wrt. the z-axis $\hat{z}$ must remain constant, $\ddot{\theta} = \dot{\theta} = 0$, which is one special case. From Eq. (3.45) we know that a constant θ must imply a constant $\dot{\phi} \equiv \Omega$, so the precession frequency is constant.

For $\ddot{\theta}$ we get from (3.54) that

$$0 = \lambda_1\dot{\phi}^2 \sin\theta\cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + MgR\sin\theta, \Rightarrow$$

$$0 = \lambda_1\Omega^2\cos\theta - \lambda_3\underbrace{(\dot{\psi} + \dot{\phi}\cos\theta)}_{=\omega_3}\Omega + MgR, \Rightarrow$$

$$0 = (\lambda_1\cos\theta)\Omega^2 - (\lambda_3\omega_3)\Omega + MgR. \tag{3.57}$$

This quadratic equation for Ω has in general two solutions. Using the standard formula and assuming ω_3 is very large, we find a small precession frequency $\Omega_- = \frac{MgR}{\lambda_3\omega_3}$ and a large precession frequency $\Omega_+ = \frac{\lambda_3\omega_3}{\lambda_1\cos\theta}$ (exercise). Ω_- agrees[10] with Eq. (3.52) in section 3.5.2. The fast possibility

[9] symmetric implies that $\lambda_1 = \lambda_2 = \lambda$

[10] It turns out the assumption of large ω_3 is equivalent to assuming a weak torque

is independent of g and corresponds to the free precession derived in section 3.2.3 (not too obviously so).

3.5.4 Nutation in the Lagrange formalism

In the discussion just completed, we had picked a very specific motion of the top by demanding $\ddot{\theta} = \dot{\theta} = 0$ from the outset. When these conditions are not perfectly met, θ will not be precisely constant and does in fact perform small oscillations around the value where it could be constant. These oscillations are called nutations.

At the same time, using again Eq. (3.45), we see that if θ may change in time also the azimuthal progress of the top ϕ, becomes more complicated. Please look at TT or GPS for more details. Here we rather directly watch

Example 36, Real dynamics of the heavy symmetric top: on Youtube. Keywords “top”, “gyroscope”, “precession”, “nutation” will find you plenty of videos. This is a brief selection of ones I liked.

- For just a fast animation of nutation, see video1 .
- Another longer animation, with explanations, see video2 .
- Now to a real experiment: video3 . Don’t worry about the counter-weight. it is just a trick to reduce the applied torque, without reducing the mass of the top/gyroscope.
- With link to online experiments (didn’t try): video4 .

Better even: Get your own toy-top and play with it. A main points that we have not yet discussed, but that is relevant for your own experiments and some of those videos is friction. It will cause the point of contact of the top on the table to move, and the overall rotation to loose energy.