Week 7

PHY 305 Classical Mechanics
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3 Motion of rigid bodies and many-bodies

After we are now equipped with two formalisms for mechanics (Newton and Lagrange), we can move to more complicated mechanical systems than you may have encountered so far. After we just looked at non-inertial frames (such as the surface of a rotating body like the earth), we now move to the dynamics of a rotating rigid body itself, and finally to the collection of many objects.

3.1 Rotational motion of rigid bodies

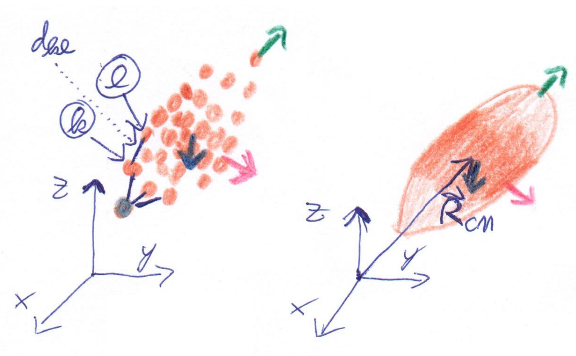
3.1.1 Rigid bodies

Consider the ellipsoidally shaped rigid body below. It could be a rugby ball or an asteroid. To place this in the context of our earlier discussions, we can

(i) Consider the rigid body as a collection of particles (see section 1.4.5). In the limit of infinitely many particles, we describe it in terms of a continuous density (see below).

(ii) To express that these mass points are rigidly attached to each other, we can use the concept of constraints (see section 2.4).

(iii) Microscopically you know the picture is in fact reasonably realistic, if you take your mass points to be the atoms making up your rigid bodies and the constraints as arising from intermolecular binding forces. But going into such a detailed picture is typically excessive.



left: (left) Rigid object decomposed as collection of constrained mass points. (right) The same object described by a continuous mass density distribution (darker=denser).

It is intuitively clear, that a rigid object such as above has 6 degrees of freedom: 3 coordinates (x,y,z) for the center of mass, and 3 more to describe its orientation in space (rotations that decide where the (Red/green/blue) axes are pointing.

We reduce the $3N$ DGFs for the individual mass points to six by using the constraint equations: $|\mathbf{r}_k - \mathbf{r}_\ell| = d_{k\ell}$, where $d_{k\ell}$ is the fixed distance between masses k and ℓ as shown in the figure. That seems to be too many constraints, but not all of these constraint equations are independent.[7] See GPS for a discussion on how these conditions constrain the number of degrees of freedom from $3N$ to 6.

Since we did not tell whether the collection of particles is constrained or not, when discussing many particles in section 1.4.5, the important definitions and results from that section carry over here. Let us repeat them, introducing also a continuum notation in terms of a mass density $\rho(\mathbf{r})$

Rigid body properties We have the total mass

$$M = \sum_k m_k = \int d^3\mathbf{r}\rho(\mathbf{r}), \tag{3.1}$$

where $\rho(\mathbf{r})$ is the mass density, the centre-of-mass positions

$$\mathbf{R}_{\mathrm{CM}} = \frac{\sum_k m_k \mathbf{r}_k}{M} = \frac{1}{M}\int d^3\mathbf{r}\, \rho(\mathbf{r})\mathbf{r}, \tag{3.2}$$

the angular momentum

$$\begin{aligned} \mathbf{L} &= \sum_k \left[\mathbf{r}_k \times m_k \mathbf{v}_k\right] = \int d^3\mathbf{r}\, \rho(\mathbf{r}) \left[\mathbf{r} \times \mathbf{v}(\mathbf{r})\right], \\ &= \mathbf{R}_{\mathrm{CM}} \times M\mathbf{v} + \sum_k (\mathbf{r}'_k \times \mathbf{p}'_k), \end{aligned} \tag{3.3}$$

where in the last part coordinates $\mathbf{r}'_k$ are in the centre-of-mass frame.

- We have included the continuum variant of all definitions, in addition to those in section 1.4.5. To convert from one to the other, use the density $\rho(\mathbf{r}) = \sum_k m_k \delta^{(3)}(\mathbf{r}_k)$, where $\delta^{(3)}(\mathbf{r})$ is a 3D Dirac delta-function (see yellow box below). Depending on the problem, the discrete or continuum definition may be more useful.
- As discussed in week 1, the first term of (3.3) can be thought of as "angular momentum of the centre-of-mass" wrt. the origin, and the second as angular momentum of the rigid body around its centre of mass.
- We had seen in Eq. (1.18) and Eq. (1.19) that the centre-of-mass motion for a rigid body is typically not that different from a point object, as long as we are able to easily calculate the

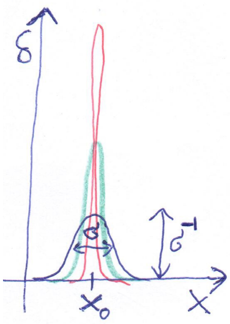
[7] The position of any one mass is fully fixed by giving its distance to three of the other masses, distances to all other masses are not required.

total external force $\mathbf{F}^{(e)} = \sum_k \mathbf{F}_k^{(e)}$ on it. Hence for the remainder of this section, we will be concerned with rotational motion only.

Dirac delta function: We define the object $\delta(x - x_0)$ such that

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x - x_0) = f(x_0), \tag{3.4}$$

for any test-function $f(x)$. In practice, you will ever only need Eq. (3.4).



left: You could loosely think of the delta-function as $= 0$ everywhere, except at x_0 where it is ∞. Clearly this definition is pathological. A better way is to think of it as the limit

$$\delta(x - x_0) = \lim_{\sigma \to 0} \frac{1}{\sqrt{\pi}\sigma} e^{-\frac{(x-x_0)^2}{\sigma^2}} \tag{3.5}$$

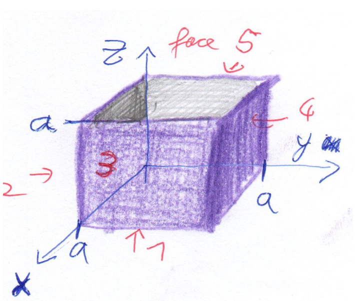
of an ever narrower Gaussian, as shown in the figure.

- The delta function is just a mathematical object, so its argument x can be any physical variable. For the 3D spatial delta-function used in the comment earlier, we use $\delta^{(3)}(\mathbf{r}_k) = \delta(x_k)\delta(y_k)\delta(z_k)$, which fulfills

$$\int d^3\mathbf{r} f(\mathbf{r})\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) = f(\mathbf{r}_0), \tag{3.6}$$

- Despite the name, the delta-function is not a genuine function, but mathematically a distribution or generalised function. That means we have to multiply it with a test-function and then integrate over it, to make sense of it, as we had seen above.

Example 26, Centre-of-mass of a cubic cup: Consider the hollow cubic cup shown below, and assume all its mass in homogeneously distributed over it surface. We want to use this example to learn how to find the centre of mass for an object given with a continuous mass distribution and to use the delta-function.



left: Cubic hollow cup with sidelength a (violet outside, grey inside), the top face of the cube is open and we assume the other cup faces are infinitely thin. First let us formulate the above as a continuous mass distribution

$$\rho(\mathbf{r}) = \varrho_0(\underbrace{\delta(z)\theta(x)\theta(a-x)\theta(y)\theta(a-y)}_{\text{bottom face } = 1} + 4 \text{ other faces}) \tag{3.7}$$

where ϱ_0 is a surface mass density, so it has units of kg/m^2. The total mass, will then be $M = 5\varrho_0 a^2$. PTO.

Example continued: We now insert Eq. (3.7) into Eq. (3.2), treating each face separately. If $\mathbf{R}_{\text{CM,k}}$ is the CM position of face k, the total CM position is then $\mathbf{R}_{\text{CM}} = \sum_{k=1}^{5} \mathbf{R}_{\text{CM,k}}/5$

$$\mathbf{R}_{\text{CM},1} = \frac{1}{M/5} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \mathbf{r}\left[\varrho_0 \delta(z)\theta(x)\theta(a-x)\theta(y)\theta(a-y)\right] \tag{3.8}$$

$$= \frac{\varrho_0}{M/5} \int_0^a dx \int_0^a dy\, \mathbf{r}\Big|_{z=0} = \frac{\varrho_0}{M/5} \begin{bmatrix} (\int_0^a dx x) \int_0^a dy \\ \int_0^a dx (\int_0^a dy\, y) \\ 0 \end{bmatrix} = \frac{\varrho_0}{M/5} \begin{bmatrix} (\frac{x^2}{2}\big|_0^a) a \\ a(\frac{y^2}{2}\big|_0^a) \\ 0 \end{bmatrix} \tag{3.9}$$

Now inserting the mass M from above, the final result for the centre of mass of the bottom of the cup is $\mathbf{R}_{\text{CM},1} = [a/2, a/2, 0]^T$, i.e. right in the middle of the bottom square as we would have guessed also without integration. Also $\mathbf{R}_{\text{CM,k}}$ for $k = 2, 3, 4, 5$ will be right in the middle of their respective squares, taking into account three dimensions. All up we can find

$$\mathbf{R}_{\text{CM}} = [a/2, a/2, 2a/5]^T. \tag{3.10}$$

Note that this lies inside the hollow of the cup, at a place $\mathbf{r}$ where $\rho(\mathbf{r}) = 0$.

Heaviside step function: (or θ function) is defined as

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{3.11}$$

- Since θ is more frequently an angle, it will be typically stated in the text if it is a Heaviside function instead.

3.1.2 Rotation about a fixed axis

We shall tackle rotation of an extended body in two stages, first considering a fixed rotation axis. This is often the case when the body is attached to some mounting point.

Since the axis is fixed, we can place the origin on the rotation axis and also choose the latter as our z-axis. Using the same definition as in section 2.9.2, Eq. (2.82), we then use the vector $\boldsymbol{\omega} = (0, 0, \omega)^T$ to describe the orientation of the rotation axis together with rotational velocity. We can now also use Eq. (2.83) as before to find the velocity of any of our mass points:

$$\mathbf{v}_k = \boldsymbol{\omega} \times \mathbf{r}_k. \tag{3.12}$$

We now proceed to evaluate the total angular momentum

$$\mathbf{L} = \sum_k \boldsymbol{\ell}_k \equiv \sum_k \left[\mathbf{r}_k \times m_k \mathbf{v}_k\right] = \sum_k m_k \left[\mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k)\right]. \tag{3.13}$$

Explicitly using $\mathbf{r}_k = [x_k, y_k, z_k]^T$ we find first $\boldsymbol{\omega} \times \mathbf{r}_k = [-\omega y_k, \omega x_k, 0]^T$ and from that

$$\boldsymbol{\ell}_k = m_k \omega [-z_k x_k,\ -z_k y_k,\ x_k^2 + y_k^2]^T. \tag{3.14}$$

Inserting this into (3.13) we now found the angular momentum and can define the

Products and moments of inertia: To relate the total angular momentum $\mathbf{L}$ to the rotational velocity as in

$$\mathbf{L} = [I_{xz}\omega, I_{yz}\omega, I_{zz}\omega]^T, \tag{3.15}$$

we define the products of inertia

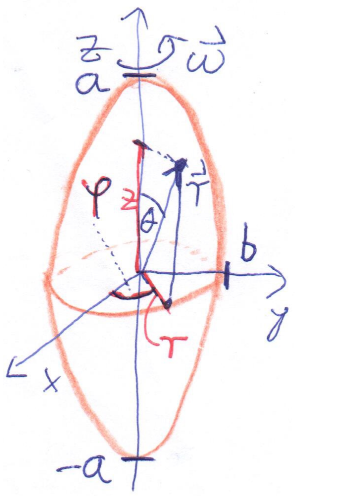
$$\begin{aligned} I_{xz} &= -\sum_k m_k x_k z_k = -\int d^3\mathbf{r}\rho(\mathbf{r})xz, \\ I_{yz} &= -\sum_k m_k y_k z_k = -\int d^3\mathbf{r}\rho(\mathbf{r})yz, \end{aligned} \tag{3.16}$$

and the moment of inertia about the z-axis

$$I_{zz} = \sum_k m_k(x_k^2 + y_k^2) = \int d^3\mathbf{r}\rho(\mathbf{r})(x^2 + y^2). \tag{3.17}$$

- As before we give all definitions in a discrete and continuum version.
- Note, that the angular momentum is in general not parallel the rotation axis. This may be counter-intuitive or surprising, however if so this is probably because the two vectors have been parallel for the examples you would have seen so far. There are many important cases where they are parallel. We shall see in examples below, that whether or not angular momentum and rotation axis are parallel depends on the symmetry of the rigid object with respect to the rotation axis.
- The moment of inertia about the z-axis depends quadratically on the distance $r_k = \sqrt{x_k^2 + y_k^2}$ of each mass point from the rotation axis.
- Moments and products of inertia depend on the chosen origin of the coordinate system. This makes sense, we already pointed out in section 1.4.1 that also the angular momentum is always to be considered with respect to a specific origin.

Example 27, Moments of inertia of a rugby ball:



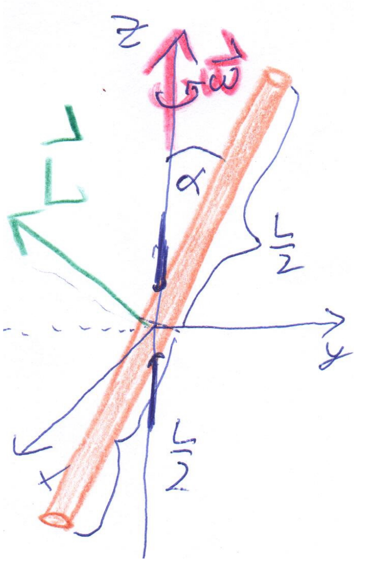
left: We model the rugby ball as an ellipsoid of constant density ρ_0 and shall use cylindrical coordinates r, z, φ. Let us find the moment of inertia about the long axis: $I_{zz} = \int d^3\mathbf{r}\rho(\mathbf{r})(x^2 + y^2)$. Since the density is constant, we can write $I_{zz} = \rho_0 \int d^3\mathbf{r}\big|_{ball}(x^2 + y^2)$, where now the integration is only over the volume of the ellipsoid. Expanding the integration in the three variables of the cylindrical coordinates we have

$$I_{zz} = \rho_0 \int_{-a}^{a} dz \int_0^{2\pi} d\varphi \int_0^{r_{\max}(z)} dr\, r \times r^2, \qquad (3.18)$$

where we can find the maximally allowed radius $r_{\max}$ from the equation of an ellipse $\left(\frac{z}{a}\right)^2 + \left(\frac{r}{b}\right)^2 = 1$, hence $r_{\max}(z) = b\sqrt{1 + (z/a)^2}$.

With that we reach $I_{zz} = 2\pi\rho_0 \int_{-a}^{a} dz \frac{b^4}{4}\left(1 + \left(\frac{z}{a}\right)^2\right)^2 = \frac{8}{15}\pi\rho_0 ab^4$. If we re-express this using the mass $M = \frac{4}{3}\pi ab^2\rho_0$ (exercise), we reach $I_{zz} = \frac{2}{5}Mb^2$.
Nextly we would want to evaluate the products of inertia, which would in principle give rise to a similar integration. However for those we can use a very important trick. Since e.g. $I_{xz} = -\int d^3\mathbf{r}\rho(\mathbf{r})xz$ depends on x, and the object is symmetric under an inversion of the x-axis, for each mass element at $+x$ there is another mass element at $-x$ that exactly cancels the contribution to the integration. We thus can directly conclude that $I_{xz} = I_{yz} = 0$ by symmetry. Hence, since the ellipsoidal rugby ball is cylindrically symmetric around the z-axis, product of inertia in other directions cancel and the angular momentum for a rotation around the z-axis is parallel to the rotation axis. This is true in general for such a symmetry.

Example 28, Moments of inertia of tilted rod:



left: Consider the rod of length L shown on the left, mounted such that it rotates around the z-axis but forms an angle α with that axis. At this snapshot in time, assume the rod is in the zy plane. We neglect the transverse size (width) of the rod and describe it with a one dimensional mass density λ, so that its total mass is $M = \lambda L$.
We then find its moment of inertia about the z-axis from

$$I_{zz} = \lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} ds (\sin(\alpha)s)^2 = \frac{\lambda \sin(\alpha)^2 L^3}{27} = \frac{M \sin(\alpha)^2 L^2}{27}. \qquad (3.19)$$

We directly see that $I_{xz} = 0$ since $x = 0$ for all elements of mass on the rod. PTO.

Example continued: However the product of inertia

$$I_{yz} = -\int d^3\mathbf{r}\rho(\mathbf{r})yz = -\lambda\int_{-\frac{L}{2}}^{\frac{L}{2}} ds(\sin(\alpha)s)(\cos(\alpha)s) = -\frac{M\sin(\alpha)\cos(\alpha)L^2}{27}. \quad (3.20)$$

is clearly non-zero. The arguments that made this expression zero in example 27 do not apply here, since the tilted stick is not symmetric under $y \leftrightarrow -y$.

We now assemble the angular momentum from Eq. (3.15) and find $\mathbf{L} = \frac{ML^2\omega}{27}[0, \sin(\alpha)\cos(\alpha), \sin(\alpha)^2]^T$, which is clearly not parallel to $\boldsymbol{\omega}$, but is tilted away from it in the yz plane. If you picture the rotation of the stick around the z-axis this also means that $\mathbf{L}$ is not constant in time [a] We know that must mean there always is a torque applied, which must be exerted by the mounting point. We will consider this in more detail in section 3.2.

[a] A little while later the stick will not lie in the zy plane, however we could define a new y' axis such that it lies in the zy' plane and fine the same result as above, so the plane containing the stick, which rotates, must also contain the angular momentum, which therefore also rotates.

The moment of inertia also appears when we calculate the

Rotational (kinetic) energy of a rigid body for roration around the z-axis, which is given by

$$T_{\text{rot}} = \frac{1}{2}\sum_k m_k\mathbf{v}_k^2 = \frac{1}{2}I_{zz}\omega^2. \quad (3.21)$$

- Proof of the second equality in (3.21):

$$T_{\text{rot}} = \frac{1}{2}\sum_k m_k\mathbf{v}_k^2 \overset{Eq.\ (3.12)}{=} \frac{1}{2}\sum_k m_k|\boldsymbol{\omega}\times\mathbf{r}_k|^2 \overset{Eq.\ (2.90)}{=} \frac{1}{2}\sum_k m_k\omega^2(x_k^2+y_k^2) = \frac{1}{2}I_{zz}\omega^2. \quad (3.22)$$

You need to do a bit of trigonometry after using Eq. (2.90) for the middle step, to see that $|\mathbf{r}_k|\sin\theta$ will give you exactly the radial distance from the rotation axis, which is $\sqrt{x_k^2+y_k^2}$.

3.1.3 Rotation about any axis

As long as the rotation axis remains fixed, we can typically choose it to be along our z-axis and use the results of the previous section. We will however see in this section, that the rotation axis often changes direction in time if it is allowed to do so. We thus have to be able to deal with more general rotations around any axis mathematically, which we work out now.

For this we step back to Eq. (3.13) and evaluate this again, this time without constraining $\boldsymbol{\omega}$ to lie on the z-axis, hence we have $\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]^T$. To sort out $\mathbf{r}_k \times (\boldsymbol{\omega}\times\mathbf{r}_k)$ we use the

BAC-CAB rule: A double cross product can be written as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{3.23}$$

and reach (suppressing subscripts k for now):

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \begin{bmatrix} (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z \\ -yx\omega_x + (z^2 + x^2)\omega_y - yz\omega_z \\ -zx\omega_x - zy\omega_y + (x^2 + y^2)\omega_z. \end{bmatrix}, \tag{3.24}$$

Based on this, we can generalize Eq. (3.16) to the

Moment of inertia tensor (or just "inertial tensor"), which we define as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}, \text{ with } I_{aa} = \sum_k m_k(b_k^2 + c_k^2), \text{ and } I_{ab} = -\sum_k m_k a_k b_k, \tag{3.25}$$

where abc are some permutation of xyz, and the middle expression is valid for identical indices and the last one for different ones only. The form (3.25) allows us to write the angular momentum through a matrix multiplication

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{3.26}$$

- For the proof of Fig. 3.26, compare with Eq. (3.24), re-instating the subscript k.
- Mathematically a "tensor" t is a map taking many vectors as an input. You can think of it as an array of values $t_{ijk...lmn}$ with many indices. For two indices this looks like component notation of a matrix M_{kl}, hence thinking of the inertia tensor just as inertia "matrix" is safe.[8].
- Check that Eq. (3.26) agrees with our earlier result (3.15) for $\boldsymbol{\omega} = (0, 0, \omega)^T$.
- The inertia tensor is symmetric, that means $I = I^T$, or $I_{xy} = I_{yx}$ etc.

[8]The only additional information implicit in the word "tensor" is a specific transformation under basis changes in your vector space

Example 29, Inertia tensor of a cube rotating around its corner: Consider a cube similar to the cup in example 26, but this time closed and filled homogeneously with a constant 3D mass density $\rho = M/a^3$. Using similar (but simpler) integration techniques as in example 27, we can find the entire inertia tensor for the cube, relative to the origin at $(0,0,0)$. See TT for details of the calculation. From (3.25), we find

$$I = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}. \tag{3.27}$$

As seen in example 28, you find from Eq. (3.26) that e.g. for $\boldsymbol{\omega} = [0,0,\omega]^T$,the angular momentum is not parallel to $\boldsymbol{\omega}$. However, for example for $\boldsymbol{\omega} = [\omega,\omega,\omega]/\sqrt{3}^T$ it would be. We again recognise the latter axis as one of higher symmetry.

We will see in the next week material that whenever angular momentum and rotational velocity are not parallel, the dynamics is substantially more complex than if they are. For that reason, it is of interest to know those rotation axis for which the two are parallel. These are called.

Example 30, Principal axes of a rigid body: For rotation about an axis through a given origin O, set up the inertia tensor I and solve the eigenvalue equation

$$I\mathbf{e}_k = \lambda_k \mathbf{e}_k. \tag{3.28}$$

The (unit) eigenvectors $\mathbf{e}_k$ are called principal axes of the rigid body. The eigenvalues λ_k are the moments of inertia for rotation around $\mathbf{e}_k$. Here $k = 1,2,3$.

- Consider a case where the rotational velocity is pointing along a principal axis, that is $\boldsymbol{\omega} = \omega\mathbf{e}$. It is easy to see by combinging Eq. (3.26) with Eq. (3.28) that these are precisely the cases where the angular momentum is parallel to the rotational velocity (exercise). By further comparison with (3.15) we then also understand the statement above, that λ_e in that case is the moment of inertia for rotation around that axis.

- Once we know all eigenvalues and eigenvectors of the inertia tensor, we can bring it into a diagonal form I', according to

$$I' = O^{-1} I O = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{3.29}$$

  where O is a matrix formed as $O = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ by writing the three eigenvectors of I beside each other as column vectors. It is an important result of linear algebra, that because the inertia tensor is symmetric $I = I^T$ and real, we can always bring it into the form (3.29) with three real eigenvalues λ_k and orthogonal $\mathbf{e}_k$. We also know that the matrix O will be an orthogonal matrix, which means $O^T = O^{-1}$. O and O^T are in fact rotation matrices, that map vectors between our original basis vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ and the basis $\{\mathbf{e}_k\}$. See more, later in section 3.2.1 and section 3.4.

- Whenever the rigid body has some clearly special axes due to its shape, the principal axes or at least some of them will point along those directions. For example the rugby ball in example 27 has $\mathbf{e}_3 = \hat{z}$.

Example 31, Principal axes of a cube around its corner: If we look again at example 3.27, we can diagonalize the inertia tensor of the cube to find

$$I' = \frac{Ma^2}{12} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}, \text{ and } O = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{3}} \\ 1 & -\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & -\frac{2}{\sqrt{3}} \end{bmatrix}. \tag{3.30}$$

We see that $\mathbf{e}_1$, which is the first column of 0 is indeed the same axis for which we had already notes in example 3.27 that for rotations around it the angular momentum will be parallel to the axis. The other two axes are here just chosen as two arbitrary axes orthogonal to $\mathbf{e}_1$.[a]

[a]The reason why this is allowed, is that the other two eigenvalues $\lambda_{2,3}$ are degenerate. If that is the case, any linear combination of the associated eigenvectors $\mathbf{e}_2' = (\cos\beta)\mathbf{e}_2 + (\sin\beta)\mathbf{e}_3$ is also an eigenvector with the same eigenvalue, so the eigenvectors of a degenerate subspace are not well defined.

We had already stated that the moment of inertia tensor and hence principal axes depend on the origin chosen. This is also made evident through (1.22), which we can use to derive the

Parallel axes theorem If we know the inertia tensor of a rigid body for an axis passing through its center of mass I_{cm}, the inertia tensor through another parallel axis is given by $I = I_R + I_{cm}$, where

$$I_R = M \begin{bmatrix} Y^2 + Z^2 & -XY & -XZ \\ -YX & X^2 + Z^2 & -YZ \\ -ZX & -ZY & X^2 + Y^2 \end{bmatrix}. \tag{3.31}$$

Here M is the total mass and $\mathbf{R}_{cm} = [X, Y, Z]^T$ in a frame that has its origin on the new axis.

- Proof: Exercise, use Eq. (3.12) and Eq. (1.22).

To conclude this section, we also generalize (3.21) to rotational motion around an arbitrary axis and find

Rotational (kinetic) energy of a rigid body for rotation $\boldsymbol{\omega}$ around any axis:

$$T_{\text{rot}} = \frac{1}{2} \sum_k m_k \mathbf{v}_k^2 = \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}. \tag{3.32}$$

- In the middle expression of (3.32) we multiply the matrix I from the left with $\boldsymbol{\omega}$ as a row vector and from the right as a column vector, in the right one we take the scalar product of $\boldsymbol{\omega}$ and $\mathbf{L}$.
- Proof: exercise, use Eq. (3.12) and Eq. (3.25).