Week 6

PHY 305 Classical Mechanics

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2.9 Mechanics in non-inertial frames

We had seen in section 1.4.2 that reference frames are either inertial or non-inertial, and Newton's equations are only valid in the former. Nonetheless one may want to work out physics in noninertial frames, particular when sitting in one, such as an accelerating vehicle or the surface of our rotating earth.

2.9.1 Straight acceleration

Consider the two frames $\mathcal{S}$ (train station) and $\mathcal{S}''$ (accelerating train) shown in section 1.4.2, but extending our picture to 3D such that the instantaneous velocity of $\mathcal{S}''$ seen in $\mathcal{S}$ is $\mathbf{V}(t)$ and its acceleration constant $\mathbf{A} = \dot{\mathbf{V}}(t)$.

Let's consider the apple of mass m at position $\mathbf{r}(t)$ dropped in the cartoon in section 1.4.2. When viewed in the inertial frame $\mathcal{S}$, we know it is described by Newton's law $m\ddot{\mathbf{r}}(t) = \mathbf{F}$, where $\mathbf{F}$ is the sum of all forces acting on it (for now gravity).

We now denote all variables in the accelerating frame $\mathcal{S}''$ by double-primes (the $''$ are <u>not</u> spatial derivatives in this section). We know that the velocity of an object in the original (unmoving) frame, $\dot{\mathbf{r}}$, is the velocity as seen in the moving frame $\dot{\mathbf{r}}''$, plus the velocity $\mathbf{V}$ of that moving frame itself:

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}'' + \mathbf{V}. \tag{2.77}$$

Up to now, it has not been important that the frame $\mathcal{S}''$ is accelerating. However when we take one more time derivative

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}'' + \dot{\mathbf{V}} = \ddot{\mathbf{r}}'' + \mathbf{A}, \tag{2.78}$$

which directly gives us

Newton's equation in an accelerating frame

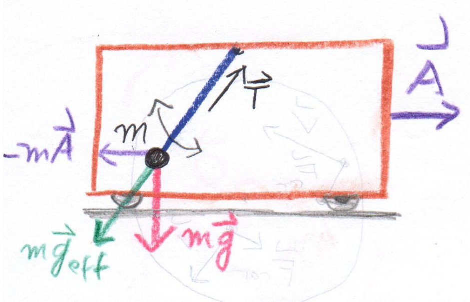
$$m\ddot{\mathbf{r}}''(t) = \mathbf{F} \underbrace{-m\mathbf{A}}_{=\mathbf{F}_{\text{in}}} \tag{2.79}$$

where $\mathbf{F}$ is the same forces as before, $\ddot{\mathbf{r}}''$ the acceleration felt in the accelerating frame and $\mathbf{F}_{\text{in}}$ labels the inertial force.

- The inertial force arises from the frame transformation only, not from some additional fundamental interaction. For that reason it is called a fictitious force. The inertial force is what presses you into the seat in an accelerating car or plane.
- You can see from (2.79) that the apple dropped in the accelerating train would indeed move slightly sideways.
- The discussion above and the following ones are phrased in terms of Newtonian mechanics instead of Lagrangian mechanics, you may wonder why after we just went through all the effort to establish Lagrange. The reason is that it would be almost too simple using Lagrange, so in order to carefully introduce frame dependent concepts such as the inertial force above, it is more instructive to stick to Newton. We will see at the end of this part, in section 2.9.3, how efforts get simplified with Lagrange.

Instead of the apple, let's consider a

Example 19, Pendulum in an accelerating train:



left: Train with acceleration $\mathbf{A}$ containing a pendulum. At rest in the accelerating frame, the tension of the string $\mathbf{T}$ will be balanced by the vector sum of the gravitational force $m\mathbf{g}$ and the inertial force $-m\mathbf{A}$, which causes a tilted equilibrium position of the pendulum as shown.

Using (2.79), we can write the equation of motion of the pendulum bob as

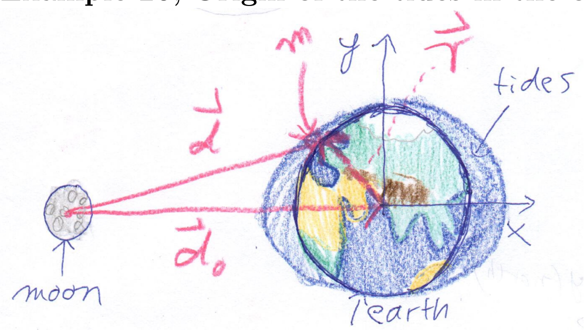
$$m\ddot{\mathbf{r}}''(t) = \mathbf{T}(\mathbf{r}) + m\mathbf{g} - m\mathbf{A}, \tag{2.80}$$

where $\mathbf{T}$ is the tension force exerted by the string. We see that we can combine gravity and the inertial forces as $m\mathbf{g} - m\mathbf{A} = m\mathbf{g}_{\text{eff}}$ into a new effective gravitational force. It just points in a slightly different direction than the real one, and has a slightly different strength. However we can just use it in the usual solution of the pendulum and thus infer its dynamics at this point. See TT for more details.

Example continued: Importantly, this example shows that there is no practical way to distinguish a gravitational force from an inertial force. This forms one of the starting points of the theory of general relativity, the principle of equivalence. It requires that the parameter m entering both forces is exactly the same. Note that while doing Newtonian physics, there is no reason for them to be the same. However they are experimentally constrained to be the same up to a fraction 10^{-15}.

For the next example we zoom into the Kepler problem of section 2.8 where we now consider the orbiting bodies to be extended objects. At every given moment, the force exerted by one of the two orbiting bodies onto the other causes an acceleration, hence when fixing the frame in one of those bodies for a short time, the frame is accelerated as in the example above[6]. We can now use Eq. (2.79) to explain the

Example 20, Origin of the tides in the ocean:



left: Sketch of earth moon system. $\mathbf{d}_0$ points from the centre of the moon to the centre of the earth, $\mathbf{d}$ to a drop of the ocean of mass m at the indicated position. The blue halo of the earth indicates the bulge of the tides, pointing towards the moon and opposite from it.

We consider three forces acting on the drop, the gravitational forces by the moon $\mathbf{F}_{moon}$, the earth $\mathbf{F}_{\text{earth}}$ and the inertial force $\mathbf{F}_{\text{in}}$, and neglect any other forces:

$$\mathbf{F}_{\text{moon}} = -GM_m m\frac{\mathbf{d}}{d^3}, \quad \mathbf{F}_{\text{earth}} = m\mathbf{g}, \quad \mathbf{F}_{\text{in}} = GM_m m\frac{\mathbf{d}_0}{d_0^3}, \tag{2.81}$$

where M_m is the mass of the moon. The inertial force $\mathbf{F}_{\text{in}}$ arises here because we work in the reference frame of the earth, which is accelerated. The earth is attracted by the moon and thus feels an acceleration $M_e\mathbf{A} = -GM_m M_e\frac{\mathbf{d}_0}{d_0^3}$.
Let us combine the pieces involving the mass of the moon into

$$\mathbf{F}_{\text{tid}} = -GM_m m\left(\frac{\mathbf{d}}{d^3} - \frac{\mathbf{d}_0}{d_0^3}\right). \tag{2.82}$$

By looking at the signs and vector directions, you can convince yourself that this points towards the moon for $\mathbf{d}$ on the side of the earth close to the moon, and away from moon on the side farther to the moon. This gives rises to two outwards bulges of the ocean surface, as sketched in the figure, and thus to high tides twice per day.

- There are additional separate tides due to the sun, which are not a lot weaker.

[6]Technically the direction of acceleration changes over one orbital period, but for now we consider only times much shorter than that.

- Note, tides also act on land, which however resists the effect more since land is a solid. Nonetheless it has noticeable consequences: the friction due to tides of the crust can eventually slow down a planetoids rotation about its own axis such that it always faces the same side to its heaviest companion. This is called "tidal locking" and has happened to our moon, which is why it always points the same side towards earth.

2.9.2 Rotating reference frames

We now zoom even further into the Kepler problem, sitting ourselves, as we are, onto the surface of the earth. While for most practical purposes our seat appears to define an inertial frame, it isn't in fact. We know the earth rotates round its axis, so we are dealing with a rotating frame, which is again non-inertial, but of a different character than the linearly accelerated frame that we saw in section 2.9.1.

A useful concept for the description of anything involving rotations is the

Angular velocity vector

$$\boldsymbol{\omega} = \omega \mathbf{u}, \tag{2.83}$$

where ω is the rotational velocity and $\mathbf{u}$ a unit vector along the rotation axis.

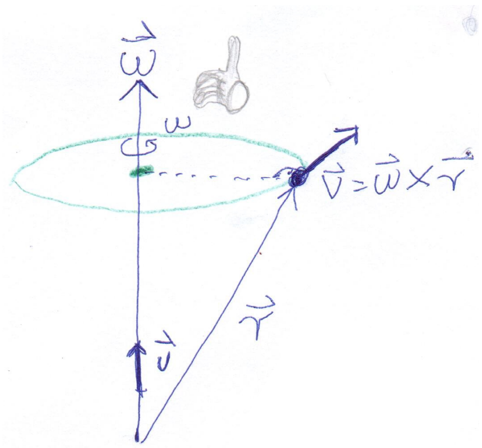
- To specific the direction of $\mathbf{u}$ we use the right hand rule, thumb $\mathbf{u}$, curved fingers direction of motion. See figure below.
- If the origin of our coordinate system is on the rotation axis, the motion due to rotation of a point at $\mathbf{r}$ is given by

  $$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{2.84}$$

  see figure below. In fact this argument works for the rate of change of any vector $\mathbf{e}$ that is fixed in the rotating frame, it evolves as $\dot{\mathbf{e}} = \boldsymbol{\omega} \times \mathbf{e}$, when viewed from the fixed frame.
- We can use Eq. (2.84) to show that rotational velocities add up: If we have three frames, where the relative velocity of 1 and 2 is $\mathbf{v}_{12}$ and that of 2 and 3 $\mathbf{v}_{23}$, we know that the relative velocity of frame 1 and 3 will be $\mathbf{v}_{13} = \mathbf{v}_{12} + \mathbf{v}_{23}$. Using this, if now instead 2 is rotating relative to 1 with $\boldsymbol{\omega}_{12}$ and 3 is rotating relative to 2 with $\boldsymbol{\omega}_{23}$, then from (2.84)

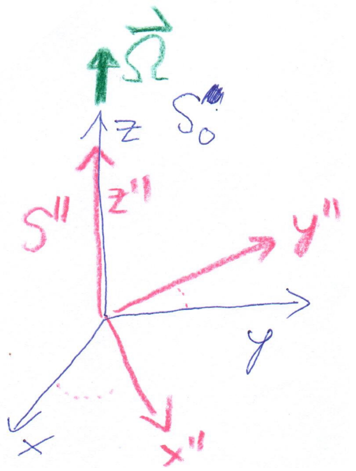
  $$\boldsymbol{\omega}_{13} \times \mathbf{r} = \boldsymbol{\omega}_{12} \times \mathbf{r} + \boldsymbol{\omega}_{23} \times \mathbf{r} = (\boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23}) \times \mathbf{r}, \tag{2.85}$$

  hence $\boldsymbol{\omega}_{13} = \boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23}$.



left: Sketch of rotation axis $\mathbf{u}$ and angular velocity vector $\boldsymbol{\omega}$, with resultant motion of a point at $\mathbf{r}$, subject to this rotation, according to Eq. (2.84).

Let us know specify what we mean by rotating frame, using the figure below:



left: We display a static frame $\mathcal{S}_0$ with cartesian axes x, y, z as usual. Relative to this, the frame $\mathcal{S}''$ is rotating with angular velocity $\boldsymbol{\Omega}$ (green). We choose $\boldsymbol{\Omega}$ along $\mathbf{e}_z$ to draw a clearer picture of how the x and y axes rotate.
We can explicitly write down the coordinate transformation shown, as

$$
\begin{aligned}
x'' &= x\cos(\Omega t) + y\sin(\Omega t),\\
y'' &= y\cos(\Omega t) - x\sin(\Omega t),\\
z'' &= z. \qquad (2.86)
\end{aligned}
$$

As in section 2.9 we again know that Newton's equation takes the usual form in the fixed frame $m\ddot{\mathbf{r}}(t) = \mathbf{F}$, and want to now derive which form it takes in the rotating frame.

First, we can show from the statement $\dot{\mathbf{e}} = \boldsymbol{\omega} \times \mathbf{e}$ after Eq. (2.84) the following

Transformation of time derivatives between frames:

$$
\left(\frac{\partial \mathbf{Q}}{\partial t}\right)_{\mathcal{S}_0} = \left(\frac{\partial \mathbf{Q}''}{\partial t}\right)_{\mathcal{S}''} + \boldsymbol{\Omega} \times \mathbf{Q}'', \qquad (2.87)
$$

where $\mathbf{Q}$ is an arbitrary time-dependent vector, and the frames and angular velocity $\boldsymbol{\Omega}$ are defined in the figure above. $(\cdots)_{\mathcal{S}_0}$ implies a time-derivative as seen in frame $\mathcal{S}_0$.

Proof: Exercise, then check solution in TT.

We now start from Newton's equation in the fixed frame

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathcal{S}_0} = \mathbf{F}. \tag{2.88}$$

Using (2.87) we can write firstly $\left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{r}''}{dt}\right)_{\mathcal{S}''} + \boldsymbol{\Omega}\times\mathbf{r}''$, then apply it a second time to get:

$$\begin{aligned}\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathcal{S}_0} &= \left(\frac{d}{dt}\right)_{\mathcal{S}''}\left[\left(\frac{d\mathbf{r}''}{dt}\right)_{\mathcal{S}''} + \boldsymbol{\Omega}\times\mathbf{r}''\right] + \boldsymbol{\Omega}\times\left[\left(\frac{d\mathbf{r}''}{dt}\right)_{\mathcal{S}''} + \boldsymbol{\Omega}\times\mathbf{r}''\right] \\ &= \ddot{\mathbf{r}}'' + 2\boldsymbol{\Omega}\times\dot{\mathbf{r}}'' + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}''). \end{aligned}\tag{2.89}$$

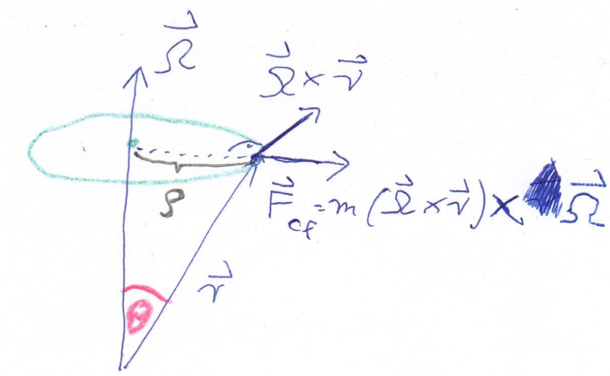
We have used here that the rotation axis and rotational velocity are time independent, i.e. $d\boldsymbol{\Omega}/dt = 0$. Inserting into (2.88), we finally reach

Newton's equation in the rotating frame

$$m\ddot{\mathbf{r}}'' = \mathbf{F} + \underbrace{2m\dot{\mathbf{r}}''\times\boldsymbol{\Omega}}_{\mathbf{F}_{\mathrm{cor}}}\underbrace{-m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}'')}_{\mathbf{F}_{\mathrm{cf}}}. \tag{2.90}$$

As before, double primes indicate that we are in the rotating frame $\mathcal{S}''$ and are not spatial derivatives. We have highlighted two new types of inertial forces, the Coriolis force $\mathbf{F}_{\mathrm{cor}}$ and the centrifugal force $\mathbf{F}_{\mathrm{cf}}$.

- Same as the inertial force in Eq. (2.79), the Coriolis force and the centrifugal force arise from the frame transformation only, hence they are also fictitious forces. We will look at their consequences in some examples below.
- You can convince yourself with multiple right-hand rules and the vector diagram on the left, that the centrifugal force points always radially outwards from the rotation axis, see figure in example below. By using the rule (2.91) below, we see that the magnitude of the centrifugal force is $|\mathbf{F}_{\mathrm{cf}}| = m\Omega^2\rho$, where ρ is the distance of $\mathbf{r}''$ from the rotation axis, see figure below.
- The centrifugal force is a manifestation of Newton's first law, that things want to go straight in the absence of forces. Since undergoing a rotation is not straight, you have to overcome the centrifugal force in order to keep an object on a rotating motion.



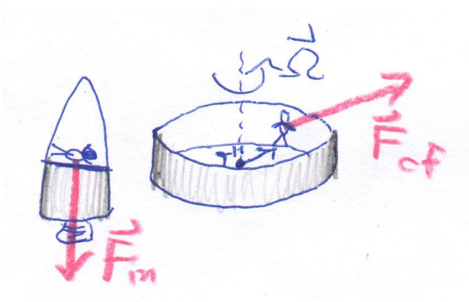
left: Vector diagram for centrifugal force.

Magnitude of cross product: We can show that

$$|\mathbf{a} \times \mathbf{b}| = ab \sin(\theta), \tag{2.91}$$

where θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

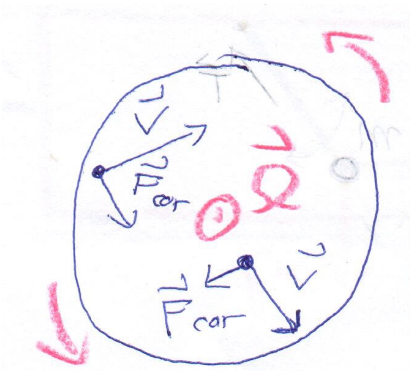
Example 21, Centrifugal force in astronaut training:



left: The centrifugal force can be used to generate strong forces/acceleration for a long time in a small spatial volume. Astronauts during rocket launch have to deal with acceleration of at least $3g$, with $g = 9.81\ m/s^2$, giving rise to an inertial force as in (2.79). You can see from Eq. (2.90), that sitting on the wall of a cylinder rotating at an angular velocity $|\boldsymbol{\omega}| = 2.5/s$ (which means a full cycle every $T = 2\pi/\omega \approx 2.5$s) at a distance $|\mathbf{r}''| = 5$m from the rotation axis will generate an outwards centrifugal force of similar magnitude.

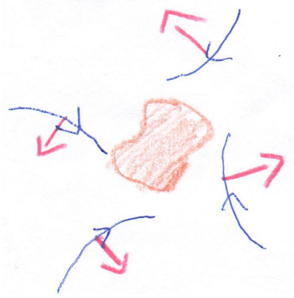
- Again using rules for cross products, we se that the Coriolis force $\mathbf{F}_{\rm cor} = 2m\dot{\mathbf{r}}'' \times \boldsymbol{\Omega}$ always points perpendicular to the velocity. It vanishes for motion that is parallel to the rotation axis, see again Eq. (2.91). In contrast to the centrifugal force, it also vanishes for objects that are not moving.

Example 22, Walking on a turntable:



left: Suppose you are walking on the disk shown on the left, which is rotating as shown (pink). Thus the frame rotation axis $\boldsymbol{\Omega}$ points out of the paper. At the positions indicated by $\bullet$ and walking into direction $\mathbf{v}$ the Coriolis force will act as shown. It will always point at right angles to your velocity. The Coriolis force takes care of the fact that motion that is on a straight line with constant velocity in the nonrotating frame, must appear curved sideways in the rotating frame, see e.g. this video or this .

Example 23, Cyclones:



left: Huge planetary scale storms are generated by air (blue arrows) moving radially into a low pressure region (brown). While doing so, the Coriolis force (pink) acts on the air, for all streams of air with the same relative orientation to the velocity.

- Another famous example involving the Coriolis force is Foucault's pendulum, essentially like example 8 but treating that in 2D and including the rotation of the earth. Due to the Coriolis force, the plane of oscillation of the pendulum will rotate, essentially due to the earth "rotating underneath it" (movie).

2.9.3 Non-inertial frames in the Lagrange formalism

(i) Lagrange's equations can be applied in non-inertial frames, as long as the (ii) Lagrangian is originally formulated in an inertial frame. The reason for (ii) is that we constructed the Lagrangian such that the output of Lagrange formalism reproduces Newton's equation as given in inertial frames. The reason for (i) is that the action principles does not care about coordinates, so we can use it in non-inertial frames as well. Let's see how this works in two examples:

Example 24, Constant acceleration: Let us denote the location of the accelerating train by $x(t)$ and the position of a person inside by $q(t)$. We want to use q as our generalized coordinate. Since we have to set up the Lagrangian in an inertial frame, we do so in $\mathcal{S}_0$ and write $\mathcal{L} = \frac{1}{2}m(\dot{x}(t) + \dot{q}(t))^2$ for a 1D free particle. However we can now evaluate Lagrange's equation (2.29) directly in the accelerating frame and find

$$\frac{\partial \mathcal{L}}{\partial q} = 0, \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{d}{dt}m(\dot{x}(t) + \dot{q}(t)) = m\ddot{x}(t) + m\ddot{q}(t). \tag{2.92}$$

and hence $m\ddot{q}(t) = -m\mathbf{A}$, which is the same as (2.79).

Example 25, Rotating frame: We again want to use the Lagrangian for a free particle $\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}(t)^2$ in the static frame, this time in 3D. The task is now to express this in a rotating frame $\mathcal{S}''$. Originally we had $\mathbf{r} = [x, y, z]^T$, we now convert this to coordinates $\mathbf{r}'' = [x'', y'', z'']^T$ using (2.86). By simple substitution, we find

$$\mathcal{L} = \frac{1}{2}m(\dot{\mathbf{r}}'' + \mathbf{\Omega} \times \mathbf{r}'')^2. \tag{2.93}$$

e.g. for $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$ this gives $\mathcal{L} = \frac{1}{2}m\big[(\dot{x}'' - \Omega y'')^2 + (\dot{y}'' + \Omega x'')^2 + \dot{z}^2\big]$. PTO

Example continued: Evaluating the parts of the Lagrange equation we find:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}''}\right) = m(\ddot{\mathbf{r}}'' + \mathbf{\Omega}\times\dot{\mathbf{r}}''), \tag{2.94}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{r}''} = m[\dot{\mathbf{r}}''\times\mathbf{\Omega} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\dot{\mathbf{r}}'')]. \tag{2.95}$$

While the first equations is a relatively straightforward application of the chain rule, the second one is more tricky, see mathbox below. Combining the two equations above into the Lagrange equation, we reproduce (2.90) as claimed.

Component notation for cross-product: Sometimes dealing with equations such as (2.95) directly in vector form can be tricky, particularly when cross products are involved. Often it helps to revert to component notation. For the cross-product $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, we can write

$$c_i = \sum_{jk}\epsilon_{ijk}a_j b_k, \tag{2.96}$$

where ϵ_{ijk} is called Levi-Civita symbol or perfectly anti-symmetric tensor. Its indices i, j, k can take the values $1, 2, 3$ or x, y, z for the three dimensions of space, and $\epsilon_{123} = 1$. Whenever two indices are equal, it is zerol, $\epsilon_{ijj} = 0$. Whenever you swap two indices, it changes sign, e.g. $\epsilon_{132} = -1$. A useful consequence is that the ϵ tensor remains constant under a cyclic change of indices, e.g. $1 = \epsilon_{123} = \epsilon_{231} = \epsilon_{312}$.

Using this, we can now write the starting point of (2.95) as

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial r''_n} &= \frac{\partial}{\partial r''_n}\frac{1}{2}m\sum_i(\dot{r}''_i + \sum_{jk}\epsilon_{ijk}\Omega_j r''_k)^2 \\ &= m\sum_i(\dot{r}''_i + \sum_{jk}\epsilon_{ijk}\Omega_j r''_k)\frac{\partial}{\partial r''_n}(\sum_{\ell m}\epsilon_{i\ell m}\Omega_\ell r''_m) \\ &= m\sum_i(\dot{r}''_i + \sum_{jk}\epsilon_{ijk}\Omega_j r''_k)(\sum_{\ell m}\epsilon_{i\ell m}\Omega_\ell\delta_{nm}). \end{aligned}\tag{2.97}$$

Here we have used the Kronecker delta symbol $\delta_{nm} = 1$ for $n = m$ and $\delta_{nm} = 0$ for $n \neq m$. Then

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial r''_n} &= m\sum_{i\ell}\underbrace{\epsilon_{i\ell n}}_{=\epsilon_{ni\ell}}\dot{r}''_i\Omega_\ell + m\sum_{ijk\ell}\epsilon_{ijk}\Omega_j r''_k\ \epsilon_{i\ell n}\Omega_\ell) \\ &= m\sum_{i\ell}\epsilon_{ni\ell}\dot{r}''_i\Omega_\ell + m\sum_{i\ell}\underbrace{\epsilon_{ni\ell}}_{=-\epsilon_{n\ell i}}\Omega_\ell\left(\sum_{jk}\epsilon_{ijk}\Omega_j r''_k\right). \end{aligned}\tag{2.98}$$

Now comparing back to (2.96) we can revert back to vector notation to reach directly (2.95).

- It appears the inertial forces in the rotating frame emerge somewhat easier in the Newtonian formalism. However an important conceptual advantage of Lagrange seen here, is that while Newton's equations have changed form from (1.5) to (2.90) when going to the inertial frame, Lagrange's equation still reads $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}''}\right) - \frac{\partial\mathcal{L}}{\partial\mathbf{r}''} = 0$, completely unchanged!!!