Week 5

PHY 305 Classical Mechanics

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2.8 Two-body central force problem

As an application of the Lagrangian formalism, we will tackle the problem of two bodies interacting with central forces with it.

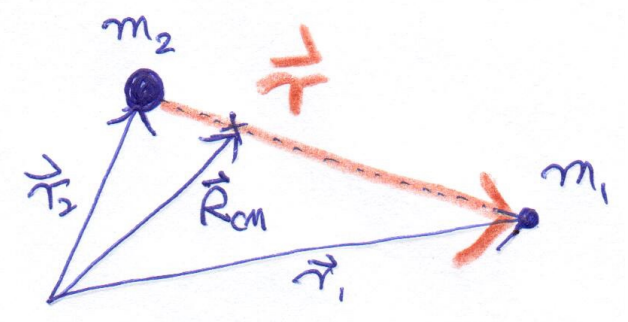
2.8.1 Coordinates and Lagrangian of the two-body problem

By two body problem we refer to the closed system of two (point-like) objects of mass m_1 and m_2 at locations $\mathbf{r}_1$ and $\mathbf{r}_2$ that mutually interact through forces $\mathbf{F}_{12}$, $\mathbf{F}_{21}$ (see section 1.4.5), but are not subject to any further external force.

left: This seemingly simplified settings actually applies to at least two very important scenarios: The interactions of planets with stars, or moons with planets and those of electrons with nuclei.

While in either case there are typically more than two objects (i.e. many planets, other molecules), frequently two of them will be a lot closer together or one of them much more important, so that it is a very good approximation to neglect all others. Of course, subatomic interaction have to be treated quantum mechanically, but doing that rigorously highly benefits from the contents of this section. We will also see one example where a classical treatment of the atom becomes a good approximation.

From the position of the two objects, we can first define further coordinates as shown below:



left: A small mass m_1 near a large mass m_2, with drawing of the centre of mass as well as relative coordinate vectiors.

Instead of $\mathbf{r}_1$ and $\mathbf{r}_2$, we can change coordinates to the centre of mass position

$$\mathbf{R}_{\rm CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \text{ with } M = m_1 + m_2, \tag{2.54}$$

which we had already seen in Eq. (1.17), and the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

The section title states that we wish to restrict ourselves to central force problems, which means that the forces are derived from a

Central potential, which depends only on the magnitude of the relative position

$$V(\mathbf{r}_1, \mathbf{r}_2) = V(|\mathbf{r}|). \tag{2.55}$$

- Well known central potentials are the gravitational potential

$$V_{\rm grav} = -\frac{Gm_1m_2}{|\mathbf{r}|}, \tag{2.56}$$

  and the Coulomb potential

$$V_{\rm coul} = -\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{r}|}. \tag{2.57}$$

- The forces can then be found from it as $\mathbf{F}_{12} = -\boldsymbol{\nabla}_{\mathbf{r}_1}V(|\mathbf{r}|)$, $\mathbf{F}_{21} = -\boldsymbol{\nabla}_{\mathbf{r}_2}V(|\mathbf{r}|)$.

The Lagrangian in the direct coordinates of the two objects takes the form

$$\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|). \tag{2.58}$$

The fact that the potential only depends on the relative coordinate, suggest the use of $R_{\rm cm}$ and $\mathbf{r}$ instead, as generalized coordinates[5] You can convince yourself that $\mathbf{r}_1 = \mathbf{R}_{\rm CM} + \frac{m_2}{M}\mathbf{r}$ and $\mathbf{r}_2 = \mathbf{R}_{\rm CM} - \frac{m_1}{M}\mathbf{r}$, using which we can convert the kinetic energy to the new coordinates:

$$\begin{aligned} T &= \frac{1}{2}m_1\left[\dot{\mathbf{R}}_{\rm CM} + \frac{m_2}{M}\dot{\mathbf{r}}\right]^2 + \frac{1}{2}m_2\left[\dot{\mathbf{R}}_{\rm CM} - \frac{m_1}{M}\dot{\mathbf{r}}\right]^2 \\ &= \frac{1}{2}\left(M\dot{\mathbf{R}}_{\rm CM}^2 + \frac{m_1m_2}{m_1+m_2}\dot{\mathbf{r}}^2\right). \end{aligned} \tag{2.59}$$

The second term motivates the definition of the

Reduced mass μ of two massive objects as

$$\mu = \frac{m_1m_2}{m_1+m_2}. \tag{2.60}$$

[5] We will not label them with q in this section.

- The reduced mass is always smaller than either m_1 or m_2.
- If one object is much heavier, say $m_2 \gg m_1$, the reduced mass is almost equal to the mass of the lighter object $\mu \approx m_1$. This applies quite well to both problems given for motivation, the planet orbiting the star, and the electron orbiting the nucleus.
- For equal mass objects $m_1 = m_2 = m$ we see that $\mu = m/2$.

We are finally in a position to write the

Lagrangian of the two-body problem

$$\mathcal{L} = \underbrace{\frac{1}{2}M\dot{\mathbf{R}}_{\mathrm{CM}}^2}_{\mathcal{L}_{\mathrm{cm}}} + \underbrace{\left(\frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(|\mathbf{r}|)\right)}_{\mathcal{L}_{\mathrm{rel}}}, \tag{2.61}$$

which decomposes as indicated into a centre-of-mass and a relative Lagrangian.

2.8.2 Equations of motion of the two-body problem

We now evaluate the Lagrange equations from (2.61). The one for $\mathbf{R}_{\mathrm{CM}}$ reads:

$$M\ddot{\mathbf{R}}_{\mathrm{CM}} = 0 \text{ which means } \dot{\mathbf{R}}_{\mathrm{CM}} = const. \tag{2.62}$$

This is as in example 14, where we saw that if $\mathcal{L}$ does not contain a coordinate (here $\mathbf{R}_{\mathrm{CM}}$), the corresponding momentum is conserved. Thus the centre of mass will just move in a straight line, at whatever velocity it had initially.

The second Lagrange equation is not much more difficult, we find

$$\mu\ddot{\mathbf{r}} = -\boldsymbol{\nabla}_{\mathbf{r}} V(|\mathbf{r}|). \tag{2.63}$$

We recognise this as mathematically equivalent to Newton's equation for a single particle of mass μ experiencing a potential $V(|\mathbf{r}|)$. Since the centre of mass motion is trivial, we have thus successfully removed all two-body character from the two-body problem, and are now effectively dealing with a single-body problem.

We can make this more formal, by saying we work in the centre-of-mass frame (see section 1.4.2), defined by $\mathbf{R}_{\mathrm{CM}}$ being the origin. In that reference frame, $\dot{\mathbf{R}}_{\mathrm{CM}} = 0$, hence $\mathcal{L}_{\mathrm{CM}} = 0$ and we really only have the Lagrangian of a single body problem.

2.8.3 Angular momentum in the two-body problem

We also see that the situation of example 16 applies here and hence the net angular momentum $\mathbf{L}$ is conserved. From the definition in section 1.4.5 and considering $\mathbf{L}$ in the centre-of-mass-frame

suggested above, we have

$$\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = \mathbf{r} \times \mu \dot{\mathbf{r}}. \tag{2.64}$$

The second equality is true in the entre-of-mass frame and left as an exercise. We see that all angular momentum is carried by the relative coordinate only, and involves the reduced mass only.

Importantly, you know that for the cross-product $\vec{a} = \vec{b} \times \vec{c}$, the vector $\vec{a}$ is automatically orthogonal to both, $\vec{b}$ and $\vec{c}$. All vectors that are orthogonal to a fixed vector form a 2D plane. Looking at (2.64), we thus deduce that at all times $\mathbf{r}$ and $\dot{\mathbf{r}}$ lie in a plane orthogonal to $\mathbf{L}$, so the motion of the two-body problem can be reduced to two dimensions! Let us pick $\mathbf{L}$ to be along the z-axis, then motion happens in the x, y plane.

Since we have a central potential, a useful choice of coordinates in that 2D plane are polar coordinates r, φ, see section 1.4.6. By inserting Eq. (1.25) into Eq. (2.61) we can find (exercise):

$$\mathcal{L}_{\text{rel}} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r). \tag{2.65}$$

We can see that it does not contain φ, and hence as discussed in section 2.7 the Lagrange equation (2.29) for φ will give us a conservation law:

$$\frac{\partial}{\partial \dot{\varphi}} \mathcal{L}_{\text{rel}} = \mu r^2 \dot{\varphi} = const = \ell, \tag{2.66}$$

where we recognize $\ell = |\mathbf{L}|$ as the (z-component of the) angular momentum. The Lagrange equation for r gives us

$$\begin{aligned} \mu \ddot{r} &= -\frac{\partial}{\partial r} V(r) + \mu r \dot{\varphi}^2 \\ &= -\frac{\partial}{\partial r} V(r) + \frac{\ell^2}{\mu r^3} = -\frac{\partial}{\partial r} V_{\text{eff}}(r). \end{aligned} \tag{2.67}$$

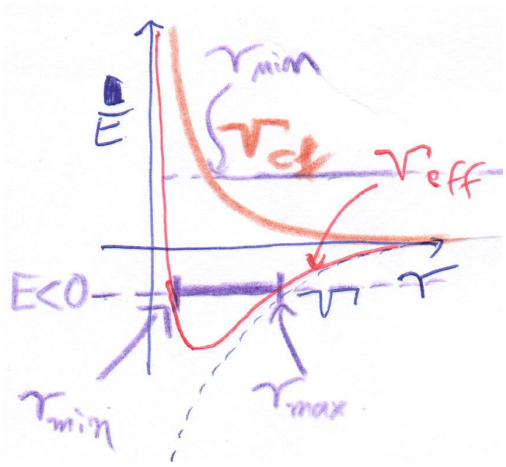
In the first step, we have eliminated $\dot{\varphi}$ in favor of r using (2.66), which brings in the angular momentum ℓ as a constant. In the final step, we have defined an effective potential for the radial motion

$$V_{\text{eff}}(r) = V(r) + V_{cf}(r) = V(r) + \frac{\ell^2}{2\mu r^2}. \tag{2.68}$$

The motivation for this name, is that the motion in the coordinate r now is simply that of a particle of mass μ in just one dimension, subject to the potential $V_{\text{eff}}(r)$. To understand the origin of the second term in (2.68), let us go to (2.67) and re-write the part $\frac{\ell^2}{\mu r^3}$ as

$$F_{cf} = \frac{\ell^2}{\mu r^3} = \frac{\mu v_\phi^2}{r}, \tag{2.69}$$

where v_ϕ is the azimuthal part of the velocity. We recognize (2.69) as the modulus of the centrifugal force. Thus we can think of the second term in (2.68) as " a potential for the centrifugal force".



left: Consider now a drawing of the effective potential $V_{\rm eff}(r)$ (red), decomposed into the real potential $V(r)$ (2.56) and the centrifugal term $V_{cf}(r)$. The centrifugal term pushes the particle outwards as you would expect. It is larger, the larger the angular momentum (see Eq. (2.68)).

Since we have reduced the dynamics with Eq. (2.67) to one dimensional motion in the potential $V_{\rm eff}(r)$, the energy conservation law $E = \frac{1}{2}\mu\dot{r}^2 + V_{\rm eff}(r)$ applies.
Hence for $E < 0$ the particle/planet cannot move the violet region between $r_{\rm min}$ and $r_{\rm max}$ and we have a bounded orbit. In contrast, for $E > 0$ arbitrarily large r is possible. This corresponds to particle scattering or a comet in an unbounded orbit.

Note that through section 2.8.1-section 2.64 we have achieved a quite dramatic simplification of our problem, from originally the six dimensions contained in $\mathbf{r}_1$ and $\mathbf{r}_2$, down to just one radial coordinate r.

2.8.4 Kepler orbits

We could now solve Eq. (2.67) with the potential (2.56) to find $r(t)$ as a function of time, but it is instructive to find $r(\varphi)$ in order to draw a picture of the orbit, as shown in the figure below. However if you imagine the orbital motion of a planet in time, r, φ and t are all linked, so we can re-convert (2.67) from differential equation in terms of t into one in terms of φ.

The detailed calculation is described e.g. in [TT] and I urge you to go through it since it contains many useful tricks. Here I only describe the summary. The two tricks used are the substitution $u = 1/r$ and differential

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot{\varphi}\frac{d}{d\varphi} = \frac{\ell}{\mu r^2}\frac{d}{d\varphi} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}, \tag{2.70}$$

where we have used (2.66) for the third equality.

We now use (2.70) to replace the time-derivative in Eq. (2.67) and reach

$$\frac{d^2}{d\varphi^2}u(\varphi) = -u(\varphi) + \frac{Gm_1m_2\mu}{\ell^2}. \tag{2.71}$$

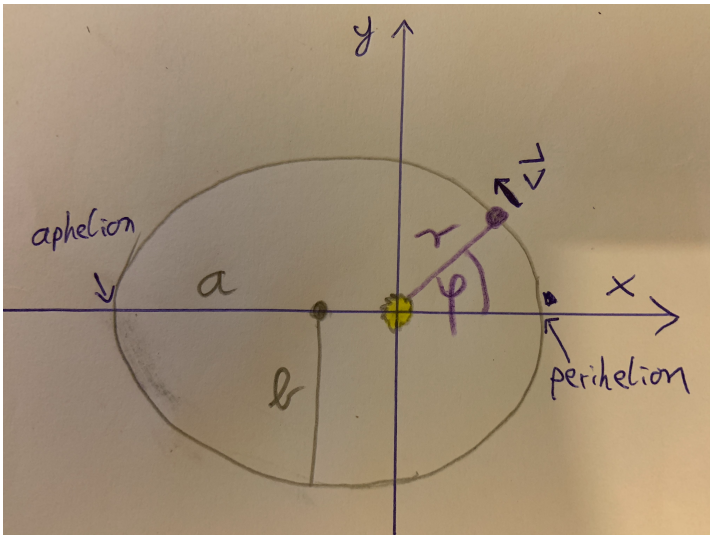
It is very useful that the second term is just a constant, since it allows the definition $w(\varphi) = u(\varphi) - \frac{Gm_1m_2\mu}{\ell^2}$ resulting in the differential equation $w''(\varphi) = -w(\varphi)$, which you know is solved by $w(\varphi) = A\cos(\varphi - \delta)$ for some constants A and δ. We remove δ by redefinition of φ, undo the definitions of w and u to get back to r and finally arrive at our

Kepler orbit

$$r(\varphi) = \frac{r_0}{1 + \epsilon \cos(\varphi)}, \tag{2.72}$$

with eccentricity ϵ. $r_0 = \ell^2/(Gm_1m_2\mu)$ is related via $r_0 = a(1-\epsilon^2)$ with the semi-major axis a.

- The new parameter eccentricity ϵ that occurs in (2.72) came from the integration constant A seen earlier, instead of A we can directly choose ϵ. The eccentricity is crucial for the type of orbit we get.
- For $0 < \epsilon < 1$, the orbit is shown in the figure below and takes the form of an ellipse, with the sun at one of the focal points. You can show this by converting (2.72) back into cartesian coordinates. That calculation gives the semi major axis $a = r_0/(1-\epsilon^2)$ and semi minor axis $b = r_0/\sqrt{1-\epsilon^2}$. We see that for $\epsilon = 0$ the orbit becomes a circle, and for $\epsilon \to 1$ it the ellipse becomes very elongated.



left: The Kepler orbit of, e.g. the earth around the sun. In terms of the figure for V_{eff} shown earlier, $r_{\min} = r_0/(1+\epsilon)$ is reached at the point marked perihelion, and $r_{\max} = r_0/(1-\epsilon)$ is reached at the point marked aphelion.

- For $\epsilon \geq 1$ we get unbounded orbits. Again, by converting Eq. (2.72) into cartesian coordinates, you can see that for $\epsilon = 1$ the orbit forms a parabola and for $\epsilon > 1$ it is hyperbolic in shape (see wikipedia "hyperpola").
- Since both, the parabola and hyperbola describe motion of the object where $r \to \infty$ is possible, we can directly tell from our drawing of V_{eff} above, that these must apply to energy $E \geq 0$ while the bounded orbits are for $E < 0$.

We shall now confirm the latter statement more explicitly by looking at the links between energy, angular momentum and eccentricity, which also help us understanding some qualitative features of the Hydrogen atom treated quantum mechanically.

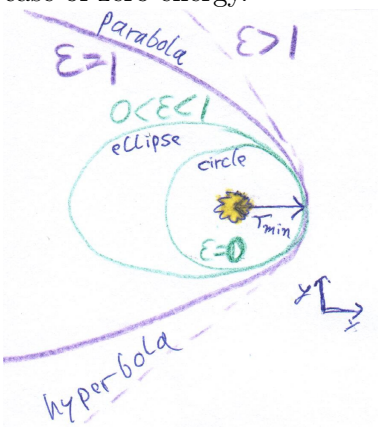
2.8.5 Energy, angular momentum and eccentricity

If we want to relate eccentricity to energy, we can go to the figure of V_{eff} above. We see that the energy equals the effective potential energy exactly at the inner turning point $E = V_{\text{eff}}(r_{\text{min}}$. Inserting $r_{\text{min}} = r_0/(1-\epsilon)$ from above into (2.68) using $V = V_{\text{grav}}$ (2.56), we can write

$$\begin{aligned} E &= -G\frac{m_1 m_2}{r_{\text{min}}} + \frac{\ell^2}{2\mu r_{\text{min}}^2} \\ &= \frac{(Gm_1m_2)^2\mu}{2\ell^2}(\epsilon^2 - 1). \end{aligned} \tag{2.73}$$

We also had to use $r_0 = \ell^2/(Gm_1m_2\mu)$.

We can now confirm the statement above, that the bounded orbits with $\epsilon < 1$ must have negative energy, while $\epsilon > 1$ have positive energy. The parabolic orbit with $\epsilon = 1$ corresponds exactly to the case of zero energy.



left: Some exemplary bounded and unbounded orbits, for fixed r_{min} but varying eccentricity ϵ, forming the geometrical shapes stated.

An important consequence of Eq. (2.73), is that for fixed energy E, the orbit still can have a variety of different angular momenta ℓ if we adjust the eccentricity accordingly. This has important consequences for atomic physics, that are encoded in the quantum results that you already know:

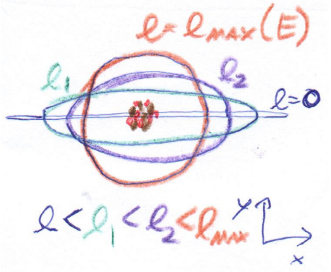
Example 18, Energy and angular momentum in the Hydrogen atom [a] **:** By swapping the gravitational potential (2.56) for Coulomb's potential (2.57), we can find a result such as (2.73) for a fictitious classical Hydrogen atom. We then have

$$E = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m_e}{2\ell^2}(\epsilon^2 - 1). \tag{2.74}$$

Now let's look at this result for fixed energy E but varying ℓ. Firstly note that $\ell \to 0$ must go with $\epsilon \to 1$. At $\epsilon \approx 1$ the Kepler ellipse becomes very very stretched, almost like a straight line. See section 1.4.1: a straight line through the nucleus would indeed have $\mathbf{L} = 0$. We see from (2.74) that at fixed E, as we lower ϵ towards zero, the angular momentum is increasing.

[a] Feel free to skip this is example until you did the Hydrogen atom in PHY303/304, but may already be able to understand it from PHY106.

Example continued: It thus takes its maximum value $\ell_{\max}$ at $\epsilon = 0$ corresponding to a circular orbit.



left: Four different angular momenta of the classical Hydrogen atom at fixed energy E, starting with $l = 0$ up to the maximum value $\ell_{\max}$.

We can write equation (2.74) for $\epsilon = 0$ as

$$-E = \frac{R_y h^2}{4\pi^2 \ell_{\max}^2} = R_y \frac{\hbar^2}{\ell_{\max}^2}, \tag{2.75}$$

where we have used the Rydberg constant $R_y = m_e e^4/(8\epsilon_0^2 h^2)$. In the hydrogen atom $E = -R_y/n^2$ and $\ell = \hbar\sqrt{l(l+1)}$, where n is the principal quantum number and l the angular momentum quantum number. You also know that $0 \leq l < n$, hence $\ell_{\max} = \hbar\sqrt{(n-1)n}$. You can now see that (2.75) gives you exactly the same relation for $\ell_{\max}$ if we approximate $n \approx (n-1)$. This might be OK, for e.g. Rydberg atoms with $n \gg 10$.

2.8.6 Additional conservation law

It turns out, in addition to conservation of energy, centre of mass momentum and angular momentum, the Kepler problem has one more conserved quantity, the

Lenz-Runge vector, defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu G m_1 m_2 \frac{\mathbf{r}}{r}. \tag{2.76}$$

- $\mathbf{A}$ always points to the Perihelion of the orbit. Its conservation thus indicates that the orientation of the Kepler orbit ellipse does not change in time, or that that the orbit perfectly closes on itself.
- For the proof that it is conserved see e.g. GPS chapter 3. We see that it is only conserved if we have a $1/r^2$ Force law, or $1/r$ potential, such as is the case for a perfect classical gravitational or Coulomb potential. When the force law deviates even slightly, an orbit no longer perfectly closes on itself. It has been one of the first observations in support of the general theory of relativity, that the planet Mercury shows a slight Perihelion advance, which can be traced back to GR causing minor deviations of the gravitational force from an $1/r^2$ Force law.