Week 4

PHY 305 Classical Mechanics
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We already got reminded in section 1.4 and the tutorials that one of the most important and practically useful principles of physics are conservation laws. It is another nice feature of the Lagrangian formalism that it eases finding new conservation laws.

2.7 Symmetries and conservation laws

Let us begin by formally defining a

Constant of the motion (or conserved quantity): A function $F(q_n, \dot{q}_n, t)$ is called a constant of the motion if

$$\frac{dF}{dt} = \sum_n \left(\frac{\partial F}{\partial q_n}\dot{q}_n + \frac{\partial F}{\partial \dot{q}_n}\ddot{q}_n \right) + \frac{\partial F}{\partial t} = 0, \tag{2.42}$$

for $q_n(t)$ that follow from Lagrange's equation.

- This is nothing new for you. You had used it all the time, we just wrote it down a little more formally.

Now let's (re-)find a few constants of the motion. Firstly, it is very easy to see the principle of

Conservation of generalised momentum If the Lagrangian does not actually depend on one of the generalised coordinates q_n, we call that coordinate cyclic or ignorable. We can directly see from Eq. (2.29), that the generalised momentum $\frac{\partial \mathcal{L}}{\partial \dot{q}_n}$ associated with that coordinate is then conserved (does not change in time).

Example 14, Conservation of linear momentum: Consider the Lagrangian for the free particle $\mathcal{L}_{\text{free}} = \frac{1}{2}m\dot{q}^2$ versus harmonic oscillator $\mathcal{L}_{\text{sho}} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$. In the free case, the Lagrangian is independent of q so the momentum $\frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} = p$ is conserved. In the second case, the Lagrangian depends on q so the momentum is not conserved.

The fundamental link between "what the Lagrangian depends on " (=symmetries) and conserved quantities is made very strong in Noether's theorem (Emmy Noether, 1918).

2.7.1 Noether's theorem

Let us first specify what we mean by symmetries of the Lagrangian.

Continuous symmetry of the Lagrangian. Suppose we have a continuous map $q_n(t) \to Q_n(s,t)$ for $s \in \mathbb{R}$, with $Q_n(0,t) = q_n(t)$. We can also call this a co-ordinate transformation. The map is called a continuous symmetry of the Lagrangian if

$$\frac{\partial}{\partial s}\mathcal{L}(Q_n(s,t), \dot{Q}_n(s,t), t) = 0. \tag{2.43}$$

- Colloquially, this means $\mathcal{L}$ does not depend on whatever change s does to your coordinate. If it confuses you now, wait for the examples below.
- Note that we deal with symmetries involving the coordinates for the moment only, thus in $Q_n(s,t)$, the inputs s and t are independent, in particular s does not depend on time. It is however possible to formulate relativistic versions of Noether's theorem that also involve time (not done here).
- We can now formulate

Noether's theorem For each continuous symmetry $Q_n(s,t)$ of the Lagrangian, there exists a constant of the motion. This constant of the motion is given by

$$\sum_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \frac{\partial Q_n}{\partial s}\bigg|_{s=0} \tag{2.44}$$

and called the Noether charge.

Proof: We can write

$$\frac{\partial}{\partial s}\mathcal{L} = \frac{\partial}{\partial s}\mathcal{L}(Q_n(s,t), \dot{Q}_n(s,t), t) = \sum_n \left[\frac{\partial \mathcal{L}}{\partial Q_n}\frac{\partial Q_n}{\partial s} + \frac{\partial \mathcal{L}}{\partial \dot{Q}_n}\frac{\partial \dot{Q}_n}{\partial s}\right]. \tag{2.45}$$

Then, starting with (2.43), we have

$$0 = \frac{\partial}{\partial s}\mathcal{L}\Big|_{s=0} = \sum_n \left[\frac{\partial \mathcal{L}}{\partial q_n}\frac{\partial Q_n}{\partial s}\Big|_{s=0} + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\partial \dot{Q}_n}{\partial s}\Big|_{s=0}\right]$$

$$\stackrel{Eq.\ (2.29)}{=} \sum_n \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right)\frac{\partial Q_n}{\partial s}\Big|_{s=0} + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\partial \dot{Q}_n}{\partial s}\Big|_{s=0}\right]$$

$$= \frac{d}{dt}\left(\sum_n \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\partial Q_n}{\partial s}\Big|_{s=0}\right]\right). \tag{2.46}$$

By $f(s)\Big|_{s=0}$ in the first step we mean $f(s=0)$. In the last line we see that the Noether charge (2.44) is a constant of the motion, proving the theorem.

All this has been very abstract so far, it should become clear with three quite fundamental examples:

2.7.2 Fundamental origin of conservation laws

Noether's law allows us to link all the major conservation laws that you know (energy, momentum, angular momentum) to symmetries of space-time.

Example 15, Momentum conservation: is linked to translation invariance. To see this, we consider the example of a bunch of point particles, with the Lagrangian

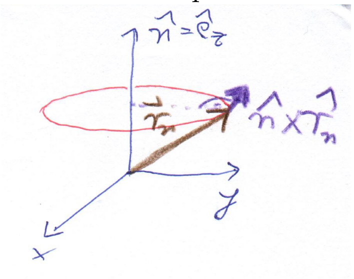
$$\mathcal{L} = \frac{1}{2}\sum_n m_n \dot{\mathbf{r}}_n^2 - \frac{1}{2}\sum_{nk} V(|\mathbf{r}_n - \mathbf{r}_k|). \tag{2.47}$$

Here $V(|\mathbf{r}_n - \mathbf{r}_k|)$ is the interaction potential between particles n and k, that depends only one the separation between them.
Now this Lagrangian will totally unchanged if we move all particles equally by an amount s into the direction $\hat{\mathbf{n}}$. We can formalize this is as $\mathbf{Q}_n(s,t) = \mathbf{r}_n(t) + s\hat{\mathbf{n}}$. You can insert this into (2.47) and see that (2.43) is fulfilled, hence this is a continuous symmetry of the Lagrangian. It is called translation invariance.
We can now find the Noether charge related to this symmetry

$$\sum_n \frac{\partial \mathcal{L}}{\partial \dot{r}_n}\cdot\frac{\partial \mathbf{Q}_n}{\partial s}\Big|_{s=0} = \sum_n m_n \dot{\mathbf{r}}_n \cdot \hat{\mathbf{n}} = \mathbf{P}\cdot\hat{\mathbf{n}}. \tag{2.48}$$

Since this is true in this example for any direction $\hat{\mathbf{n}}$, we see that the conserved quantity associated with translation invariance is the complete total momentum $\mathbf{P} = \sum_n m_n \dot{\mathbf{r}}_n$, see section 1.4.5. If the system was only translation invariant in a specific direction, e.g. $\hat{\mathbf{n}} = \hat{\mathbf{e}}_x$, only the corresponding component of the total momentum would be conserved.

Example 16, Angular momentum conservation: is linked to rotation invariance. In the same example as for example 15, let's look at an infinitesimal rotation.



left: infinitesimal rotation for infinitesimal s

$$\mathbf{Q}_n(s,t) = \mathbf{r}_n(t) + s\hat{\mathbf{n}} \times \mathbf{r}_n(t). \tag{2.49}$$

Exercise: why is this an infinitesimal rotation?

Example continued: The Lagrangian is again invariant under this rotational symmetry, see assignments, hence we identify the Noether charge for this as:

$$\sum_n \frac{\partial \mathcal{L}}{\partial \dot{r}_n} \cdot \frac{\partial \mathbf{Q}_n}{\partial s}\bigg|_{s=0} = \sum_n \mathbf{p}_n \cdot (\hat{\mathbf{n}} \times \mathbf{r}_n) \overset{Eq.\ (2.51)}{=} \sum_n \hat{\mathbf{n}} \cdot (\mathbf{r}_n \times \mathbf{p}_n) = \hat{\mathbf{n}} \cdot \mathbf{L}. \tag{2.50}$$

Similarly to the argumentation in the previous example, we see that the total angular momentum $\mathbf{L} = \sum_n \mathbf{r}_n \times \mathbf{p}_n$ is conserved if the Lagrangian is symmetric under rotation around any axis $\hat{\mathbf{n}}$, or some component of it might be conserved if there are only special symmetry axis (i.e. the z-axis for cylindrical symmetry).

Vector identity 1: For three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, you can show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{2.51}$$

Finally we have of course

Example 17, Energy conservation: is linked to time-translational invariance. If the Lagrangian does not explicitly depend on time, we can say it is invariant under $t \to t + s$. In that case we also have that $\partial \mathcal{L}/\partial t = 0$, from which we can show the conserved quantity:

$$H = \sum_n \dot{q}_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \mathcal{L}. \tag{2.52}$$

proof: exercise. When evaluating (2.52) explicitly for the Lagrangian (2.47) we find

$$H = \frac{1}{2}\sum_n m_n \dot{\mathbf{r}}_n^2 + \frac{1}{2}\sum_{nk} V(|\mathbf{r}_n - \mathbf{r}_k|), \tag{2.53}$$

which we recognise as the total energy of the system (section 1.4.5). It is called here also "Hamiltonian" (function), you have seen the operator version in quantum physics, and you will see the Hamiltonian function again in this course in much more detail in chapter 4.
We have thus found that energy conservation is related to time translation invariance.