Week ③

PHY 305 Classical Mechanics
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Before showing the Lagrange equations in ND, we will first discuss constraints, which are what makes Lagrange's formalism and Hamilton's principle really more powerful than Newton's equations. In section 1.4 we had seen how to deal with N point particles that are completely free to move in 3 dimensions. This has realistic physics applications (e.g. a bunch of astrophysical bodies, or microscopical atoms), but frequently would be too simple. On many length scales in between these two, bodies may have to be dealt with as extended (with shape and size), and/or their motion may be constrained.

2.4 Constrained systems and generalized coordinates

You have already seen some constrained systems, for example the

Example 8, Simple pendulum:
A ball of mass m is attached to a comparatively massless rigid stick of length ℓ, attached at one end to the origin O such that the stick can rotate in the xy plane. The motion of the mass would take place in 2D in principle, however instead of using coordinates (x, y) to describe the motion, we can use for example simply the angle ϕ.
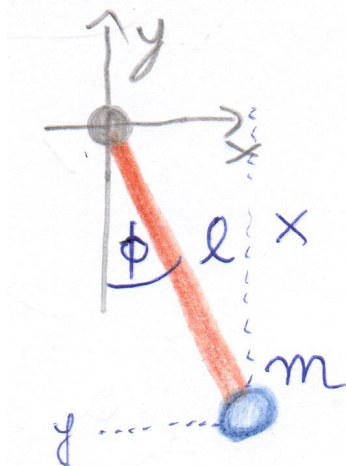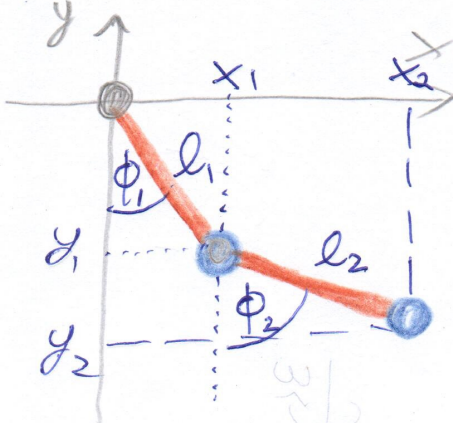

left: Sketch of rigid pendulum. Since we know the length of the stick, ϕ tells us where the ball is. The <u>constraint</u> on the ball's position is thus $\ell = \sqrt{x^2 + y^2}$. Instead of x, y, we shall see shortly that we can describe everything with the <u>generalised coordinate</u> $q = \phi$.

We can make this slightly more complicated, and go to the

Example 9, Double pendulum:
It starts with one rigid pendulum as in the example above, but now a second one is hanging attached to the first.

left: Sketch of double pendulum. Since the lengths of both sticks are fixed at ℓ_1 and ℓ_2, we need only two angles $\phi_{1,2}$ to know the positions of both balls, instead of x_1, y_1, x_2, y_2. The constraints can be written as $\ell_1 = \sqrt{x_1^2 + y_1^2}$ and $\ell_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Our generalised coordinates would be $(q_1, q_2) = (\phi_1, \phi_2)$.

These two examples motivate the definition of

Generalised coordinates For a system of N particles at positions $\mathbf{r}_n$, we call the parameters $q_1, \cdots q_M$ generalised coordinates, if all positions can be expressed as a function of these and possibly time

$$\mathbf{r}_n = \mathbf{r}_n(q_1, \cdots, q_M, t) \tag{2.19}$$

and the reverse

$$q_m = q_m(\mathbf{r}_1, \cdots, \mathbf{r}_N, t). \tag{2.20}$$

We also need M to be minimal.

- For the simple pendulum example above, we had $\mathbf{r} = [x, y] = [\ell \sin(\phi), -\ell \cos(\phi)]$, which takes the form (2.19).

- For the double pendulum example above, $\mathbf{r}_1 = [x_1, y_1] = [\ell_1 \sin(\phi_1), -\ell_1 \cos(\phi_1)]$ and $\mathbf{r}_2 = [x_2, y_2] = [\ell_1 \sin(\phi_1) + \ell_2 \sin(\phi_2), -\ell_1 \cos(\phi_1) - \ell_2 \cos(\phi_2)]$, which is again of the form (2.19).

- We see that due to the constraints, we may need a smaller number of generalised coordinates than original coordinates (which is of course the point of using them).

- Physically, of course the constraints also arise dynamically, through constraint forces $\mathbf{F}_{\text{cstr}}$. If the mass in example 8 above would want to move further out than length ℓ, the tension arising in the stick would hold it back. The tension forces can be very large, so that essentially no motion that does not fulfill $\ell = \sqrt{x^2 + y^2}$ is allowed. We shall see shortly, that this usually means that we will not have to deal with the constraint forces explicitly.

2.4.1 Types of constraints

We have seen that constraints can be helpful in reducing the number of variables needed to describe a problem, however not all constraints are equally helpful. Let us briefly mention some nomenclature regarding the

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Classification of constraints

Constraints are <u>holonomic</u>, if they can be written in the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N, t) = 0, \qquad (2.21)$$

for some function f, and <u>non-holonomic</u> if that is not possible.

They are called <u>rheonomous</u> if the constraint equation (2.21) explicitly involves time t and <u>scleronomous</u> if it does not.

- You can see that the constraints for the pendulum examples above are holonomic and scleronomous (exercise).

- An example of non-holonomic constraints, would be a bunch of molecules confined into a sphere of radius R. Their positions then are constrained by the in-equality $|\mathbf{r}_n| \leq R$, which we cannot write in the form (2.21).

2.4.2 Degrees of Freedom

Degrees of Freedom

The number of degrees of freedom (DGF), is the number of coordinates of a physical system that can be independently varied in a small physical displacements from its position.

- A particle free to move in 3D has three DGFs.

- N particles free to move would have $3N$ DGFs.

- The pendulum in the example above has one and the double pendulum two. We see that if a system is constrained it will have less DGF than the maximum number ($N_{\text{particles}} \times N_{\text{dimensions}}$)

- WARNING: In a statistical mechanics, the word "degrees of freedom" is often used slightly differently, there it would include momentum which makes it typically $2\times$ as many as discussed here.

- For <u>holonomic systems</u> the number of DGF equals the number m of <u>generalised coordinates</u> required to describe it.

For an example where DGF$\neq M$, which is therefore non-holonomic, see the following example.

Example 10, Ball rolling on a horizontal table:

left: Consider the red-green ball shown on the left, and imagine it placed on a flat table-top which shall be the xy plane. Assume it can only roll and not slip. It can now be rolled either in the x or into the y direction as shown, so it has two degrees of freedom.

However, to fully describe the situation of the ball, you have to separately record its orientation (i.e. two angles θ, ϕ). We try to indicate orientation in the sketch by drawing a ball that is red on one half and green on the other. While θ, ϕ are linked to x, y through the non-slip conditions and cannot vary fully independently, they are also not fully determined by (x, y). This is shown by the rhs diagram, where we roll the ball on the triangular path such that it reaches the same position x_0, y_0 that it started from. However, depending on the details of the path, its orientation can be changed. You can try this out easily in an experiment.

2.5 Lagrange formalism

2.5.1 Hamilton's principle for many particles and with constraints

We now proceed to the generalisation of Hamilton's principle to an N-dimensional constrained system that we promised at the end of section 2.3. It reads

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Hamilton's principle (ND) Let a complicated (possibly constrained) mechanical system be described by positions $\mathbf{R} = [\mathbf{r}_1, \cdots, \mathbf{r}_N]^T$ of N particles[a]. The path taken by the system is then such that the action

$$S = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}, t) dt \qquad (2.22)$$

is stationary on that path. Here $\mathcal{L} = T - V$ is the Lagrangian as in (2.15).

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[a]Do not confuse this with the centre of mass position, for which I write $\mathbf{R}_{\mathrm{CM}}$

Proof: To show this, we follow a similar procedure as in section 2.12. We write the real (physical) "path" taken by the system in time as $\mathbf{r}(t)$. Note that this is a very high-dimensional path, that contains all positions of all particles if there is more than one. Any (imagined) wrong path $\mathbf{R}(t)$ of the system in the vicinity of $\mathbf{r}(t)$ can then be written as $\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\epsilon}(t)$, where $\boldsymbol{\epsilon}(t)$ is some small deviation. Since we require our variational calculation to probe all paths that go from the

real $\mathbf{r}(t_1)$ to the real $\mathbf{r}(t_2)$, we impose that $\boldsymbol{\epsilon}(t_1) = \boldsymbol{\epsilon}(t_2) = 0$. While the path may be wrong, we require it to respect the constrains, i.e. if a particle is constrained to move on a 2D hilly surface within 3D, then also $\mathbf{R}(t)$ must lie on that surface.

The action (2.22) will be stationary, if its value does not change for small infinitesimal $\boldsymbol{\epsilon}(t)$. This is a similar statement to "you find the minimum (an extremal point) of a function $f(x)$, by requiring $f'(x) = 0$". Let us evaluate the difference between the action on the wrong path S and on the real path S_0, we call that $\delta S = S - S_0$. Inserting (2.22), we see that this will be just the integration in time over $\delta \mathcal{L} = \mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}, t) - \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t)$. We let the Lagrangian be

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r}, t). \tag{2.23}$$

Importantly $V(\mathbf{r}, t)$ <u>does not</u> contain potentials for the constraint forces, it describes the non-constraint forces only. We now evaluate $\delta \mathcal{L}$ by inserting $\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\epsilon}(t)$ and then using the fact that $\boldsymbol{\epsilon}(t)$ is small:

$$\delta \mathcal{L} = \frac{1}{2}m\big[(\dot{\mathbf{r}} + \dot{\boldsymbol{\epsilon}})^2 - \dot{\mathbf{r}}^2\big] - \big[V(\mathbf{r} + \boldsymbol{\epsilon}, t) - V(\mathbf{r}, t)\big]$$

$$= m\dot{\mathbf{r}} \cdot \dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} V + \mathcal{O}(\epsilon^2). \tag{2.24}$$

Now we insert this into (2.22) to get

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L} dt = \int_{t_1}^{t_2} \big[m\dot{\mathbf{r}} \cdot \dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} V\big] dt \tag{2.25}$$

In the first term, you can move the time derivative from $\boldsymbol{\epsilon}$ onto $\mathbf{r}$ using integration by parts:

$$\int_{t_1}^{t_2} \dot{\mathbf{r}} \cdot \dot{\boldsymbol{\epsilon}} dt = [\dot{\mathbf{r}} \cdot \boldsymbol{\epsilon}]_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\mathbf{r}} \cdot \boldsymbol{\epsilon} dt = - \int_{t_1}^{t_2} \ddot{\mathbf{r}} \cdot \boldsymbol{\epsilon} dt. \tag{2.26}$$

The second equality follows due to $\boldsymbol{\epsilon}(t_1) = \boldsymbol{\epsilon}(t_2) = 0$ discussed at the top. We arrive at

$$\delta S = - \int_{t_1}^{t_2} \boldsymbol{\epsilon} \cdot \big[m\ddot{\mathbf{r}} + \boldsymbol{\nabla} V\big] dt \tag{2.27}$$

Now the right path $\mathbf{r}$ follows Newton's law so $m\ddot{\mathbf{r}} = \mathbf{F}_{\text{tot}}$, where $\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{cstr}} + \mathbf{F}$ is the total force on the system (This is also a high dimensional vector $\mathbf{F}_{\text{tot}} = [\mathbf{F}_1, \cdots, \mathbf{F}_N]^t$), which we can split as shown into constraint and non-constraint forces. We now know that $\boldsymbol{\nabla} V$ gives us the non-constraint force $\boldsymbol{\nabla} V = -\mathbf{F}$. Alltogether this means:

$$\delta S = - \int_{t_1}^{t_2} \boldsymbol{\epsilon} \cdot \mathbf{F}_{\text{cstr}} dt. \tag{2.28}$$

But the constraint forces are per definition orthogonal to any deviation of the path $\boldsymbol{\epsilon}$ that is consistent with the constraint, see figure below, so $\boldsymbol{\epsilon} \cdot \mathbf{F}_{\text{cstr}} = 0$ and we have shown that $\delta S = 0$ to first order in ϵ, which means it is stationary (QED).
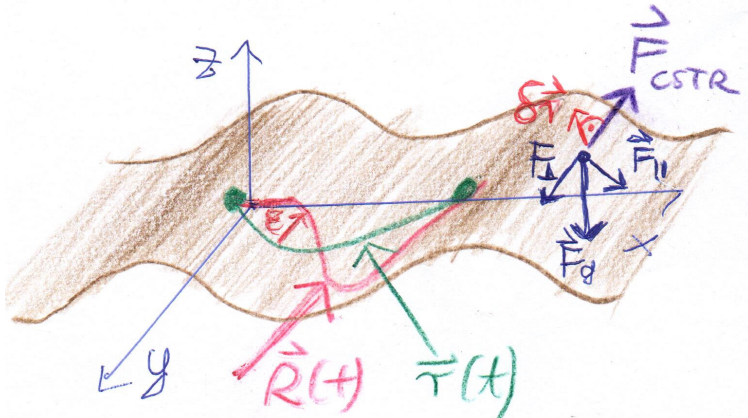
left: Sketch of dynamics constrained into the brown wavy surface. The green path would be the correct path **r**t an frictionless point-particle would take between the two green dots. The pink line is some arbitrary wrong path $\mathbf{R}(t)$ and $\boldsymbol{\epsilon}$ the difference between them at some point in time...

On the right we sketch how the gravitational force $\mathbf{F}_g$ on the blue dot can be decomposed into a force $\mathbf{F}_\parallel$ within the brown plane, and an orhogonal component $\mathbf{F}_\perp$. The latter is completely cancelled by the constraint force $\mathbf{F}_{\text{cstr}}$, which makes sure the mass stays on the brown plane.

2.5.2 Lagrange's equations

Now, because of the constraints we would like to change coordinates from $\mathbf{R}$ to a set of generalised coordinates $\mathbf{q} = [q_1, \cdots q_M]^T$, see section 2.4. A very powerful feature of Hamilton's principle is, that it is <u>independent</u> of the coordinates used [4].

We thus know from Hamilton's principle, that the action $S = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt$ is also stationary at the physical path expressed in generalised coordinates. Thus allows us now to use again the Euler-Lagrange equations (2.12) (in the multi-variate version) and directly write the

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Lagrange equations For a holonomic system with M generalized coordinates, the Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial q_n} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \qquad (2.29)$$

with Lagrangian $\mathcal{L} = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, \dot{\mathbf{q}}, t)$, where V is the potential energy for the non-constraint forces.

- We derive Eq. (2.29) from the stationarity of the action (2.22) using Eq. (2.12). For that, in (2.12) rename $x \to t$, $y(x) \to q(t)$, $y'(x) \to \dot{q}(t)$.

- Lagrange equations are surprisingly simple to set up. You find the kinetic energy T and potential energy V in your generalised coordinates and subtract them to form the Lagrangian. Then you evaluate the partial derivatives in Eq. (2.29). Importantly, <u>partial</u> derivatives wrt. q_n and $\dot{q}_n$ imply that, e.g. when calculating $\frac{\partial \mathcal{L}}{\partial q_n}$ you treat $\dot{q}_n$ as <u>independent</u> of q_n.

- Limitations: We have derived the ND action principle only for <u>holonomic</u> constraints (see

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[4] If the right path is a stationary point of S in some coordinates, it will be so also in any other coordinates.

section 2.4.1). This is not that obvious, but it subtly comes in when we have used the Euler-Lagrange equation (2.12) to go from the stationarity of (2.22) to Eq. (2.29). Recall that the multi-variate Euler-Lagrange equation require the functions entering the functional to be independent, which is not generally the case for non-holonomic constraints.

- Extensions towards non-holonomic systems are possible but difficult (See books). We had also required that the non-constraint forces are conservative. This for example excludes friction forces. Again extensions to include these are possible but difficult.

- Lagrange's equations take the same form in any coordinates system, however you have to be careful that the reference frame (see section 1.4.2) in which you originally write down the Lagrangian $\mathcal{L}$ is an <u>inertial frame</u>.

We had already seen in section 2.3 that for a simple particle in 1D, the lhs of the Lagrange equation becomes $\mathbf{F}$ and the rhs $\dot{\mathbf{p}}$. For this reason we define the

> **Generalised force and momentum** We say that $\frac{\partial \mathcal{L}}{\partial q_n}$ is component n of the generalised force and $\frac{\partial \mathcal{L}}{\partial \dot{q}_n}$ is component n of the generalised momentum.

Lagrange equations then just says that "the rate of change of the generalised momentum equals the generalised force", just as we know (without the word " generalised") for Newton's equation.

- Lagrange equations are way more easy to apply to many complicated problems than Newton's equations. This is best to see in examples, at which we look now and in the assignments. Let us first apply Lagrange's equations to the examples from section 2.4.

> **Example 11, Simple pendulum:**
> The kinetic energy is $T = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. Using $[x, y] = [\ell \sin(\phi), -\ell \cos(\phi)]$ this becomes $T = \frac{1}{2}m\ell^2[(\cos(\phi)\dot{\phi})^2 + (\sin(\phi)\dot{\phi})^2] = \frac{1}{2}m\ell^2\dot{\phi}^2$. The potential energy of the mass is $V = mgh$, where h is the height. With a bit of geometry we see $V = mg\ell[1 - \cos(\phi)]$. We can express the entire Lagrangian using our generalised coordinate $q = \phi$:
>
> $$\mathcal{L}(q, \dot{q}) = \frac{1}{2}m\ell^2\dot{q}^2 - mg\ell[1 - \cos(q)]. \tag{2.30}$$
>
> Now we calculate the parts of (2.29) and find $\frac{\partial \mathcal{L}}{\partial q} = -mgL\sin(q)$ and $\frac{\partial \mathcal{L}}{\partial \dot{q}} = m^2\dot{q}$, so that the <u>Lagrange equation for the simple pendulum</u> becomes
>
> $$m\ell^2\ddot{q} = -mg\ell \sin(q). \tag{2.31}$$
>
> Using the angular momentum $L = |\mathbf{L}|$ (see section 1.4.1) of the mass around the origin $L = m\ell^2\dot{q}$ and the torque $N = |\mathbf{N}|$ (see section 1.4.1) excerted by gravity $N = -mg\ell \sin(\phi)$, we see that this reproduces Eq. (1.6), <u>without</u> that we had to invoke any of these concepts.

- We defer an actual solution of the differential equation (2.31) as well as the double pendulum to the assignments.

Example 12, Block sliding on a wedge revisited:

Let us look at example 1 again. You had solved the question when the block hits the floor in assignment 1 using Newtonian mechanics. We now redo the problem using Lagrangian mechanics. For the we need to set up the Lagrangian, and thus require the kinetic and potential energy, in terms of the underlined generalised coordinates q_1, q_2 shown in the sketch below.
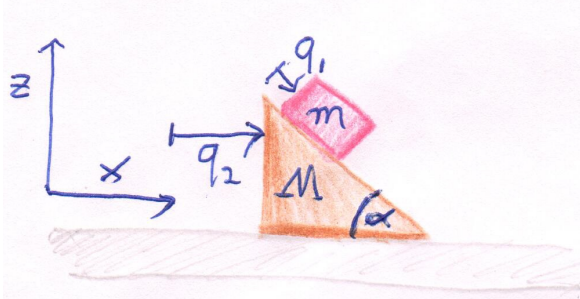
left: The same block sliding on a wedge as before, but this time with a definition of the generalized coordinates q_1, the distance of the block from its starting corner of the wedge (relative to the wedge), and q_2, the horizontal position of the wedge itself.

The cartesian coordinates $[x, z]$ of the block in terms of generalised coordinates are $[x, z] = [q_2 + q_1 \cos(\alpha), -q_1 \sin(\alpha)]$. This makes it easy to write the kinetic energy for the block

$$T_m = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m[\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos(\alpha)]. \tag{2.32}$$

We have to add the kinetic energy of the wedge $T_M = \frac{1}{2}M\dot{q}_2^2$ to T_m, to reach

$$T = \frac{1}{2}(m + M)\dot{q}_2^2 + \frac{1}{2}m[\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 \cos(\alpha)]. \tag{2.33}$$

The potential energy of the wedge does not change and we set it to zero, while that of the block is $V = mgz = -mgq_1 \sin(\alpha)$. Thus the total Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}(m + M)\dot{q}_2^2 + \frac{1}{2}m[\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 \cos(\alpha)] + mgq_1 \sin(\alpha). \tag{2.34}$$

We now write down the Lagrange equations (2.29), first for q_2:

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{d}{dt}\left(M\dot{q}_2 + m[\dot{q}_2 + \dot{q}_1 \cos(\alpha)]\right) \tag{2.35}$$

which implies that the generalised momentum for variable q_2 (in curved brackets) is constant in time:

$$M\dot{q}_2 + m[\dot{q}_2 + \dot{q}_1 \cos(\alpha)] = const. \tag{2.36}$$

You can see that this is just the conservation of momentum in the x direction. ...

Example continued:

... The equation for q_1 is:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \Leftrightarrow$$

$$mg\sin\left(\alpha\right) = \frac{d}{dt}m(\dot{q}_1 + \dot{q}_2\cos\left(\alpha\right)) = m(\ddot{q}_1 + \ddot{q}_2\cos\left(\alpha\right)) \tag{2.37}$$

We can differentiate (2.36) with respect to time to get an equation for $\ddot{q}_2$ and use it to eliminate $\ddot{q}_2$ in (2.37), finally reaching:

$$\ddot{q}_1 = \frac{g\sin\left(\alpha\right)}{1 - \frac{m\cos^2\left(\alpha\right)}{M+m}}. \tag{2.38}$$

This is a constant acceleration down the slope, so within time, the block will move $q_1(t) = \frac{1}{2}\ddot{q}_1 t^2$, thus to travel the length ℓ of the wedge it will take time $t = \sqrt{2\ell/\ddot{q}_1}$, which answers the question.

You can now compare this solution method to the one involving Newton's equation and forces in assignment 1. While for this example both are still doable, Lagrange is generally regarded more elegant and quickly becomes much more tractable for even more complicated problems.

Example 13, Bead on a spinning hooop:

As a second example, consider a bead of mass m threaded onto a hoop spinning with angular frequency ω as shown in the figure below, and visualized in movies to be found on youtube: movie1 and movie2 .
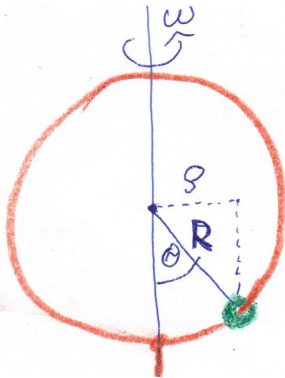
left: Sketch of hoop (brown) with radius R, rotating with angular frequency ω about the rotation axis (z-axis) shown in blue. At an angle θ as shown, a green bead is threaded on the hoop and can slide on the hoop freely. When it has an offset ρ from the rotation axis, the bead will dynamically slide under a balance of gravity and centrifugal forces.

The Lagrangian for this setup is (exercise/ see Taylor book)

$$\mathcal{L} = T - V = \frac{1}{2}mR^2[\dot{\theta}^2 + \omega^2\sin^2\left(\theta\right)] - mgR[1 - \cos\left(\theta\right)]. \tag{2.39}$$

Example continued:
The Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \Leftrightarrow$$

$$mR^2 \omega^2 \sin(\theta) \cos(\theta) - mgR \sin(\theta) = mR^2 \ddot{\theta} \Leftrightarrow$$

$$\ddot{\theta} = (\omega^2 \cos(\theta) - g/R) \sin(\theta). \qquad (2.40)$$

We shall solve this equation numerically in the assignments, however even without computer we can find equilibrium positions θ_0 of the bead (i.e. those where the angle does not change $\dot{\theta}\big|_{\theta_0} = 0$. This requires also $\ddot{\theta} = 0$, for which we can solve Eq. (2.40).
We find that either $\sin\theta_0 = 0$, which means the bead is at the top or bottom of the hoop $\theta = 0, \pi$. This places it on the rotation axis, where it does not experience a centrifugal force, so it can stay there. Alternatively $\cos\theta = g/(\omega^2 R)$. For this to have a real solution we need $\omega^2 > g/R$. This leads to the interesting situation that up to some critical rotation frequency $\omega_c = \sqrt{g/R}$, the bead will stay at the bottom, but when that frequency is exceeded it will suddenly settle at $\theta = \arccos[g/(\omega^2 R)]$, see animations in second video.

You can find a large number of additional examples in the textbooks, and we shall see a few more in assignments and tutorials.

Please read the following section because you need to know some names of things from in there, however it mainly points to advanced material and will not be exam relevant.

2.6 D'Alembert's principle (bonus material)

This figure at the end of section 2.5.1, also already indicated a "virtual displacement" $\boldsymbol{\delta r}$. This is akin to the deviation of the wrong path from the real path. We just mathematically ask what happens if the system undergoes a virtual displacement, assuming all forces remain constant. This is to be distinguished from a "real displacement" which happens in time only due to Newton's equations. From this we can define the

Virtual work $\delta W = \mathbf{F} \cdot \boldsymbol{\delta r}$.

- In equilibrium, the virtual work for all displacements is zero.

Instead of (2.22), another slightly different viewpoint is to directly jump to (2.41) and base mechanics on

d'Alembert's principle, which states that

$$0 = \boldsymbol{\delta r} \cdot \left[\dot{\mathbf{p}} - \mathbf{F} \right]. \tag{2.41}$$

where $\boldsymbol{\delta r}$ is a <u>virtual displacement</u> and $\mathbf{F}$ the non-constraint forces.

- Already the term in square brackets is zero via Newton's equations, so why we would write down (2.41) become clear only when following it futher.

- It is possible to derive Lagrange's equations (2.29) from d'Alembert's principle instead of Hamilton's principle, see books.

- However d'Alembert's principle is less widely useful in physics than variational methods, which is why we concentrated on the latter.

- One nice aspect though, is that it directly gives rise to the form of the Lagrangian as $\mathcal{L} = T - V$.