

Week 1

PHY 305 Classical Mechanics

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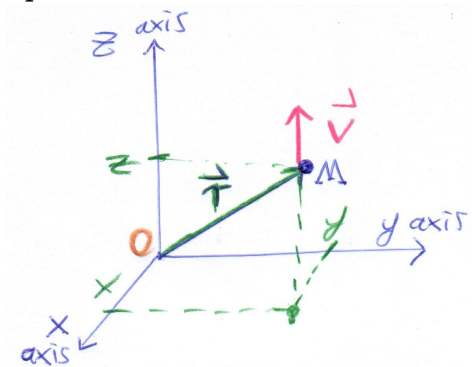
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1.4 Review of Newtonian Mechanics

The following is intentionally brief, since I assume that you have covered all or most of it in PHY101. Please let me know if it is too fast. We will assess your working knowledge of these concepts in the life-quiz session of one of the first weeks.

1.4.1 Elementary principles

Space and time



left: Let us consider a point-like particle of mass m as in the sketch on the left.

- We first decide on a coordinate system with origin O and orthogonal axes as shown. In terms of these we can express the position of the object as

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = \sum_j r_j \mathbf{e}_j. \quad (1.2)$$

As show we use **boldface** to denote vectors in this document and \vec{r} in handwritten elements. We shall use both notations above for the unit-vectors in the x , y , z directions. Finally we may also use a colum or row vector notation such as

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (1.3)$$

All these representations of \mathbf{r} makes use of Cartesian coordinates. Other coordinate system are possible, as we shall see in section 1.4.6.

- Typically the position might be a function of time $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$, which means each vector component is a function of time. From this we define the velocity as

$$\mathbf{v}(t) = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}} + v_z(t)\hat{\mathbf{z}} = \dot{x}(t)\hat{\mathbf{x}} + \dot{y}(t)\hat{\mathbf{y}} + \dot{z}(t)\hat{\mathbf{z}} = \dot{\mathbf{r}}(t). \quad (1.4)$$

As usual we shall use $\dot{f} = \frac{d}{dt}f(t)$ to denote the total time derivative of $f(t)$ with respect to t . Technically, we must decide a “co-ordinate system” for time also, at least an origin i.e. which moment do we call $t = 0$.

Mechanics of a Particle

The concepts above are sufficient to describe the motion of the particle, but not enough to describe the origin or rules of that motion. For this we require also

- The momentum $\mathbf{p}(t) = m\mathbf{v}(t)$. As indicated, unless otherwise stated we shall assume that masses do not change in time. We know that momentum is conserved, i.e. $\dot{\mathbf{p}}(t) = 0$, unless a Force acts on the particle.
- If a Force acts on the particle, the momentum may change according to

$$\mathbf{F}(t) = m\ddot{\mathbf{x}}(t) = \dot{\mathbf{p}}(t). \quad (1.5)$$

We also call $\ddot{\mathbf{x}}(t)$ the acceleration \mathbf{a} .

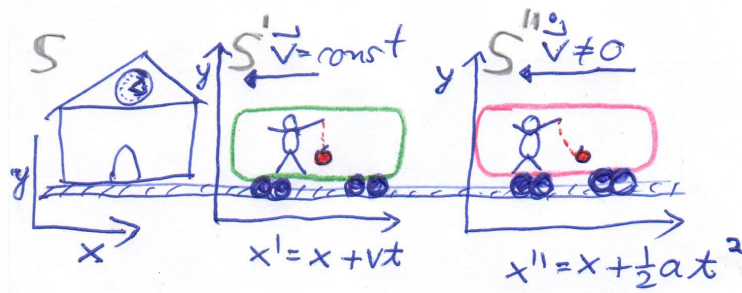
- Equally fundamental as the momentum is the angular-momentum $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t)$ with respect to the origin O .¹
- Angular momentum is conserved, unless a torque \mathbf{N} acts on the particle, which is defined as $\mathbf{N} = \mathbf{r} \times \mathbf{F}$ (and thus also taken ”with respect to the origin”).
- If a torque acts on the particle, its angular momentum changes according to

$$\mathbf{N}(t) = \dot{\mathbf{L}}(t). \quad (1.6)$$

1.4.2 Reference Frames

- A concept subtly different from a coordinate system is that of a reference frame, stating in particular the state of motion of the observer.
- For simplicity however, let’s think of a reference frame as a moving cartesian coordinate system. That means the origin O of the moving coordinate system becomes a function of time when expressed in a non-moving coordinate system. Consider the examples below.

¹It is a frequent mistake to forget that angular momentum depends on the origin of your co-ordinate system.



top: Sketch of three different reference frames. \mathcal{S} (train station), \mathcal{S}' (green train, constant velocity), \mathcal{S}'' (red train, accelerating)

- Let us refer to the three reference frames in the figure above as \mathcal{S} , \mathcal{S}' , \mathcal{S}'' as listed in the caption. A statement as in "green train" implies that the green train is stationary in the reference frame \mathcal{S}' , which implies that the latter moves with the same velocity to the left as the train does. We also indicate the transformation of the coordinate x to a new coordinate x' (or x'') in each of the coordinate systems.
- We refer to \mathcal{S} and \mathcal{S}' as inertial frame, since Newton's first law holds in them. Think of these as frames at rest or frames moving with constant velocity. You know from experience that even in a moving train, all physical laws appear unchanged.
- Different from these are non-inertial frames, namely accelerated frames such as \mathcal{S}'' or rotating frames. In these the physical laws appear different. For example when you drop an apple in an accelerating train, it would appear to move also sideways instead of just downwards.

1.4.3 Newton's laws

You know that most of the above can be summarized in

Newton's laws

- In the absence of forces, a particle in an inertial frame moves with constant velocity \mathbf{v} (law of inertia).
- For a particle of mass m , the net force (sum of all forces), is given by $\mathbf{F} = m\mathbf{a}$, see Eq. (1.5).
- If an object A exerts a force \mathbf{F}_{ab} onto an object B, then object B exerts exactly the opposite force $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$ onto object A.

1.4.4 Work and energy

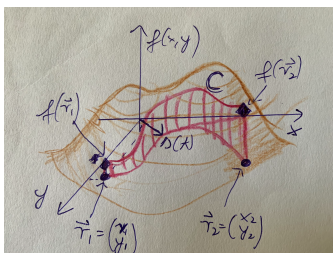
Forces may give rise to work, based on which we have the concept of energy, which is frequently more useful for the practical solution of problems.

- We define the work W_{12} done on a particle when moving it from location \mathbf{r}_1 to location \mathbf{r}_2 as

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r}. \quad (1.7)$$

This definition uses a 3D line integral, which we review/define for this occasion below:

Line-integral of scalar function: A line-integral refers to integrating over a scalar or vector valued function over a curve \mathcal{C} in a higher-dimensional space (typically in 2D or 3D). Let us look at the scalar case first, with integrand $f(\mathbf{x}) = f(x, y)$.



left: Schematic of line-integral over scalar function in 2D. Consider a 2D curve as shown, defined by the (vector valued) function $\mathbf{s}(t)$, such that $\mathbf{s}(0) = \mathbf{r}_1$ and $\mathbf{s}(1) = \mathbf{r}_2$, using $0 \leq t \leq 1$. We call $\mathbf{s}(t)$ a parametrisation of the curve and t its parameter [not necessarily time].

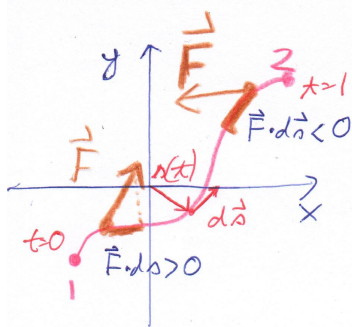
In that case, the line integral is defined as:

$$\int_{\mathcal{C}} f(\mathbf{x}) ds = \int_{t=0}^{t=1} f(\mathbf{s}(t)) |\mathbf{ds}(t)|, \quad (1.8)$$

where $\mathbf{ds}(t)$ is a tangent vector to the curve at the location $\mathbf{s}(t)$. We can find it using

$$\mathbf{ds}(t) = \frac{\partial}{\partial t} \mathbf{s}(t) dt. \quad (1.9)$$

Line-integral of vector function:



left: Schematic of line-integral over vector function in 2D. We again consider the same parametrised curve as in the scalar case, but this time the function integrated over is a vector. Suppose we know a vector \vec{F} for all points on the curve, so we can write $\vec{F}(t)$. The line integral is then defined via

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{t=1} \mathbf{F}(t) \cdot \mathbf{ds}(t) = \int_{t=0}^{t=1} \mathbf{F}(t) \cdot \frac{\partial}{\partial t} \mathbf{s}(t) dt. \quad (1.10)$$

The line integral thus consists of projecting the vector \mathbf{F} onto the curve at each point and integrating these up. Importantly, with (1.10) we reduced the 2D line integral over a vector function to a common integral that you have seen before. We shall see examples in the assignments.

- Using Eq. (1.7) to integrate along the motion of a particle $\mathbf{r}(t)$ from t_1 to t_2 (so now the curve

parameter *is* the time), we find

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \stackrel{\text{Eq. (1.5)}}{=} m \int_1^2 \dot{\mathbf{v}} \cdot d\mathbf{r} \stackrel{\text{Eq. (1.4), Eq. (1.9)}}{=} m \int_{t_1}^{t_2} \dot{\mathbf{v}} \cdot \mathbf{v} dt = \frac{m}{2} [\mathbf{v}^2(t_2) - \mathbf{v}^2(t_1)]. \quad (1.11)$$

This motivates the definition of the kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$, and we see that the work amounts exactly to the change of kinetic energy between point 1 and point 2: $W_{12} = T_2 - T_1$.

- Force fields such that the work W_{12} is independent of the path that is taken from \mathbf{r}_1 to \mathbf{r}_2 are said to be conservative. A mathematical way of stating the above, using the line-integral we just encountered, is to say:

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0, \quad (1.12)$$

where we use the symbol \oint to denote a line integral over a loop that closes on itself (i.e. $\mathbf{r}_1 = \mathbf{r}_2$).

- Whenever (1.12) holds, one can show that it is possible to write

$$\mathbf{F} = -\nabla V(\mathbf{r}), \quad (1.13)$$

i.e. the force is the gradient of a potential (energy) $V(\mathbf{r})$.

Gradient: The gradient ∇ is a vector valued derivative operator. Explicitly

$$\nabla f(\mathbf{r}) = \begin{bmatrix} \frac{\partial}{\partial x} f(\mathbf{r}) \\ \frac{\partial}{\partial y} f(\mathbf{r}) \\ \frac{\partial}{\partial z} f(\mathbf{r}) \end{bmatrix} \quad (1.14)$$

or $\nabla = \sum_k \mathbf{e}_k \frac{\partial}{\partial r_k}$.

- Using the potential, we can write Eq. (1.11) also as $W_{12} = V(\mathbf{r}_1) - V(\mathbf{r}_2) = T_2 - T_1$. We thus see that the total energy of the particle $E = T + V$ is conserved. This assumed that V does not depend on time.

1.4.5 Systems of Particles

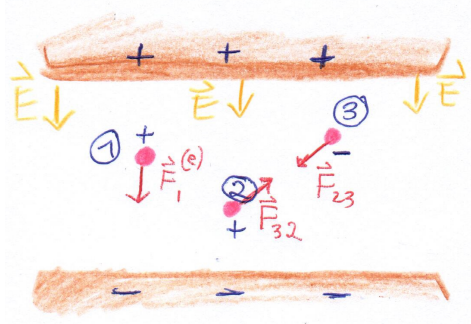
Typically a single point particle as discussed so far is not very interesting. At the very least we want a large number of point particles. We can also use these to approximately describe large extended objects. All concepts introduced so far can be straightforwardly generalized by adding a subscript k numbering the particle, e.g. instead of Eq. (1.2) we have the position

$$\mathbf{r}_k = x_k \hat{\mathbf{x}} + y_k \hat{\mathbf{y}} + z_k \hat{\mathbf{z}} = \sum_j r_{jk} \mathbf{e}_j. \quad (1.15)$$

of particle k , or its momentum \mathbf{p}_k . The momentum of one particle will then change according to

$$m_k \ddot{\mathbf{v}}_k = \dot{\mathbf{p}}_k = \mathbf{F}_k^{(e)} + \sum_{\ell} \mathbf{F}_{\ell k}, \quad (1.16)$$

where $\mathbf{F}_k^{(e)}$ is the external force and $\mathbf{F}_{\ell k}$ the force exerted by particle ℓ onto particle k .



left: An example would be a collection of electrically charged particles in a parallel plate capacitor as shown on the left.

Centre of mass position and momentum:

There are frequent examples where we want to know about the group of particles as a whole, rather than each individual one, hence we define the total mass $M = \sum_k m_k$, where m_k is the mass of particle k , and from that the centre-of-mass position

$$\mathbf{R}_{\text{CM}} = \frac{\sum_k m_k \mathbf{r}_k}{M}. \quad (1.17)$$

When we sum Eq. (1.16) over all particles, we then find:

$$M \ddot{\mathbf{R}}_{\text{CM}} = \sum_k \mathbf{F}_k^{(e)} \equiv \mathbf{F}^{(e)}, \quad (1.18)$$

which means the centre of mass responds to the net external force $\mathbf{F}^{(e)}$ only. The reason that internal forces $\mathbf{F}_{\ell k}$ have dropped out from Eq. (1.16) is Newton's third law, due to which $\mathbf{F}_{\ell k} = -\mathbf{F}_{k\ell}$ and hence $\sum_{k\ell} \mathbf{F}_{\ell k} = 0$.

Using the total momentum $\mathbf{P} = \sum_k \mathbf{p}_k$, we can write $\dot{\mathbf{P}} = \mathbf{F}^{(e)}$, which implies that without a net external force the total momentum is conserved. We also see

$$\mathbf{P} = M \dot{\mathbf{R}}_{\text{CM}}, \quad (1.19)$$

so the total momentum can be thought of being carried by the total mass, positioned at the centre-of-mass.

Angular momentum:

We can show similarly, that the net angular momentum $\mathbf{L} = \sum_k (\mathbf{r}_k \times \mathbf{p}_k)$ is changed according to the net external torque $\mathbf{N}^{(e)} = \sum_k \mathbf{r}_k \times \mathbf{F}_k^{(e)}$ (see e.g. GPS)

$$\dot{\mathbf{L}} = \mathbf{N}^{(e)} \quad (1.20)$$

While the total linear momentum can be viewed as due to motion of the centre-of mass, see Eq. (1.19), the link is more subtle for angular momentum: We first write all positions and velocities relative to the centre of mass: $\mathbf{r}_k = \mathbf{r}'_k + \mathbf{R}_{\text{CM}}$ and $\mathbf{v}_k = \mathbf{v}'_k + \mathbf{v}$, where $\mathbf{v} = d\mathbf{R}_{\text{CM}}/dt$ is the velocity of the centre of mass and $\mathbf{v}'_k = d\mathbf{r}'_k/dt$. Then

$$\begin{aligned} \mathbf{L} &= \sum_k [\mathbf{r}_k \times m_k \mathbf{v}_k] = \sum_k [(\mathbf{r}'_k + \mathbf{R}_{\text{CM}}) \times m_k (\mathbf{v}'_k + \mathbf{v})] \\ &= \sum_k (\mathbf{r}'_k \times m_k \mathbf{v}'_k) + \left(\sum_k \mathbf{r}'_k \right) \times m_k \mathbf{v} + \sum_k \mathbf{R}_{\text{CM}} \times m_k \mathbf{v}'_k + \sum_k (\mathbf{R}_{\text{CM}} \times m_k \mathbf{v}) \\ &= \sum_k (\mathbf{r}'_k \times m_k \mathbf{v}'_k) + \left(\sum_k m_k \mathbf{r}'_k \right) \times \mathbf{v} + \mathbf{R}_{\text{CM}} \times \frac{d}{dt} \left(\sum_k m_k \mathbf{r}'_k \right) + \sum_k (\mathbf{R}_{\text{CM}} \times m_k \mathbf{v}) \quad (1.21) \end{aligned}$$

We can see that $\sum_k m_k \mathbf{r}'_k = 0$ by using the definitions of \mathbf{R}_{CM} and \mathbf{r}'_k (essentially $\sum_k m_k \mathbf{r}'_k$ is the centre of mass in the frame where the centre of mass is the origin). Hence we arrive at the

Angular momentum of many particles

$$\mathbf{L} = \mathbf{R}_{\text{CM}} \times M\mathbf{v} + \sum_k (\mathbf{r}'_k \times \mathbf{p}'_k). \quad (1.22)$$

- The first term can be thought of as “angular momentum of all mass concentrated at the centre-of-mass” wrt. the origin, and the second as angular momentum of the particle collection around its centre of mass.
- See also discussion in GPS, the main purpose of replicating it here is to practice manipulations of abstract sums and cross products.

Energies:

- We of course can also generalize the energy concepts to many particles. We write the total kinetic energy

$$T = \frac{1}{2} \sum_k m_k \mathbf{v}_k^2 = \frac{1}{2} M \mathbf{v}^2 + \frac{1}{2} \sum_k m_k \mathbf{v}'_k{}^2. \quad (1.23)$$

- If we assume both internal and external forces to be conservative, we can also define a total potential

$$V = \sum_k V_k + \frac{1}{2} \sum_{k\ell} V_{k\ell}, \quad (1.24)$$

with the properties that $V_{k\ell} = V(|\mathbf{r}_k - \mathbf{r}_\ell|)$, $\mathbf{F}_{k\ell} = -\nabla_\ell V_{k\ell} + \nabla_k V_{k\ell} = -\mathbf{F}_{\ell k}$ and $\mathbf{F}_k^{(e)} = -\nabla_k V_k$.

- Note that in (1.24) symbols with different numbers of indices are different potentials.
- We then find again that the total energy $E = T + V$ using Eq. (2.33) and Eq. (1.24) is conserved. For more discussion see GPS.

1.4.6 Two-dimensional polar coordinates

Cartesian coordinates as introduced in Eq. (1.2) are the simplest, but frequently unsuitable for some problem, e.g. if that has a spherical or circular symmetry. To review how to deal with non-cartesian coordinates, we choose 2D polar coordinates r, φ . These are defined via

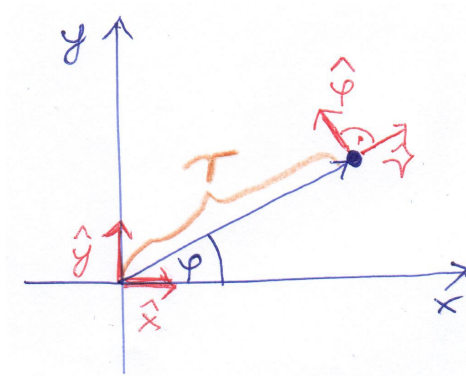
$$\left. \begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \end{aligned} \right\} \leftrightarrow \left\{ \begin{aligned} r &= |\mathbf{r}| = \sqrt{x^2 + y^2} \\ \varphi &= \arctan(y/x) \end{aligned} \right., \quad (1.25)$$

where you have to manually ensure that \arctan gives you angles $\varphi \in [0, 2\pi)$, and actually use $\text{atan2}(y,x)$

We now define the

Unit vectors of 2D polar coordinates:
radial unit vector $\hat{\mathbf{r}}$ and azimuthal unit vector $\hat{\boldsymbol{\varphi}}$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}, \quad \hat{\boldsymbol{\varphi}} = \frac{\partial}{\partial \varphi} \hat{\mathbf{r}} = \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix} \quad (1.26)$$



left: Diagram for 2D polar coordinates and their unit vectors. To understand the meaning of the unit vectors, go back to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. $\hat{\mathbf{x}}$ point in the direction that the particle moves when increasing x at constant y . Similarly $\hat{\mathbf{r}}$ points in the direction it moves when increasing r at fixed φ etc.

Using these unit vectors we can write the position in polar coordinates as simply

$$\mathbf{r} = r \hat{\mathbf{r}}. \quad (1.27)$$

Let us now move to the deceptively simple sounding task of writing down Newton's second law (1.5) in polar coordinates. The exerted force is just some unspecified vector, which we can write as $\mathbf{F} = F_r \hat{\mathbf{r}} + F_\varphi \hat{\boldsymbol{\varphi}}$ as discussed above. Let us now evaluate the acceleration $\ddot{\mathbf{x}}$, which is where it gets painful. While (1.27) looks very simple, note that now both, the coordinate $r(t)$ and the unit-vector $\hat{\mathbf{r}}(t)$ depend on time. Thus

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}(t) \hat{\mathbf{r}} + r(t) \frac{\partial}{\partial t} \hat{\mathbf{r}} = \underbrace{\dot{r}(t)}_{=v_r} \hat{\mathbf{r}} + \underbrace{r(t) \dot{\varphi}(t)}_{=v_\varphi} \hat{\boldsymbol{\varphi}}. \quad (1.28)$$

The last equality follows from Eq. (1.26). One more time differentiation and even more use of the product and chain rules, gives you

$$\mathbf{a} = \ddot{\mathbf{r}} = [\ddot{r}(t) - r(t)\dot{\varphi}(t)^2] \hat{\mathbf{r}} + [r(t)\ddot{\varphi}(t) + 2\dot{r}(t)\dot{\varphi}(t)] \hat{\boldsymbol{\varphi}}. \quad (1.29)$$

Through comparison with \mathbf{F} , we see that Newton's equation in polar coordinates take the form

$$F_r = m(\ddot{r} - r\dot{\varphi}^2), \quad (1.30)$$

$$F_\varphi = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \quad (1.31)$$

This is much more messy than in cartesian coordinates, most importantly we lose the nice fact that one vector component of the force only depends on the acceleration of that same vector component of the position. Matters get even worse in spherical polar coordinates for 3D. It is one of the main motivations behind Lagrangian and Hamiltonian mechanics that these do not get more messy in non-cartesian coordinate systems.

1.4.7 Preview: Configuration space versus phase-space

We would normally discuss phase-space only in chapter 4) on Hamiltonian mechanics, however due to the unusual course schedule in 2020 we will prepone it. It is a quite simple concept and very important for statistical mechanics (PHY306).

We had earlier described the position of N particles by N 3D vectors \mathbf{r}_n . We could write these also as a single $3N - D$ vector $\mathbf{R} = [\mathbf{r}_1, \dots, \mathbf{r}_N]^T$. The space of all possible \mathbf{R} is called configuration space of this system.

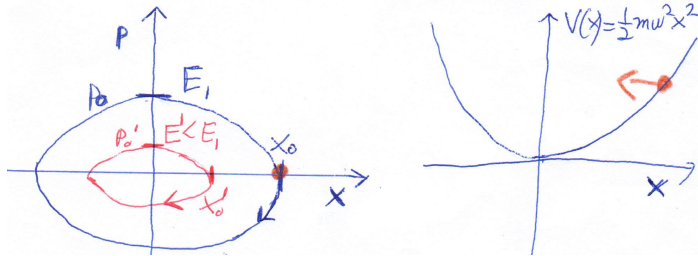
The configuration space is not really sufficient to describe the complete state of the system, since in order to arrive at a unique solution of Newton's equations (1.5), we also require all particle's velocities (see assignment and tutorial1). We thus move from configuration space to the

Phase space:

The phase space of an N particle system (without constraints) in 3D is given by the N 3D positions \mathbf{r}_n and the N 3D linear momenta \mathbf{p}_n .

- Since the \mathbf{p}_n also specify the velocities \mathbf{v}_n , once we know the “phase-space coordinate” $[\mathbf{r}_n, \mathbf{p}_n]^T$ of the system at $t = 0$, we also know them at all $T > 0$ due to (1.5).
- For N particles in 3 Dimensions, phase-space is $6N$ dimensional.
- Phase space is also useful to visualize all states available to a system given conservation laws, for example all positions and momenta consistent with energy conservation (see example below).
- It then will play a major role in statistical physics (PHY306), where a central assumption is that there are so many particles, that all states consistent with some conservation laws will be occupied.

Example 3, Phase space of harmonic oscillator: You know that a harmonic oscillator undergoes periodic motion $x(t) = x_0 \sin(\omega_0 t)$, hence $p(t) = m\dot{x}(t) = p_0 \cos(\omega_0 t)$, where $p_0 = mx_0\omega_0$. We can visualize this in phase space as shown below:



left: Simple harmonic motion looks like an ellipse in phase-space. The larger the energy, the larger the ellipse. As time goes on, the ellipse is traversed clockwise.