Week 12

PHY 305 Classical Mechanics
Instructor: Sebastian Wüster, IISER Bhopal, 2020

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4.7 Nonlinear Mechanics and Chaos

It turns out, an important criterion regarding the complexity of solutions for any given mechanical problem, is whether the equations of motion are linear or non-linear in the coordinates. Linear ones will typically give rise to relatively straightforward behavior such as oscillations, as we had seen in section 3.6.2. Nonlinear ones, can give rise to what is colloquially called "chaos" and more carefully "strong sensitivity to initial conditions". This final week section of the course is meant as only an "appetiser" for non-linear mechanics, this could easily be a standalone course (and in fact is one, PHY 411). As mention in section 1, non-linear mechanics is one of the major arenas with ongoing research in classical mechanics.

However another purpose of this section is to practice the important concept of phase-space introduced in week10, and to see some more uses for it.

4.7.1 Linearity and Nonlinearity

We already stated above, that we distinguish linear and non-linear physical systems, by whether their equations of motion are linear or non-linear in the coordinates. Let us look at examples that we had seen already:

Example 48, Linear and Non-linear systems:
One of the simplest linear systems is the simple harmonic oscillator, e.g. consider a single cart in section 3.6.1, for which we can write

$$\ddot{q} = -\omega_0^2 q, \tag{4.59}$$

which contains q only linearly.
In contrast for the simple pendulum, see example 4.22, we have upon writing a single equation of motion

$$\ddot{\phi} = -\frac{g}{l}\sin(\phi), \tag{4.60}$$

which is non-linear due to the sin function.

Example continued: In this case however, once could linearise the problem by considering only small deviations of the pendulum from equilibrium $\phi \approx 0$, for which we can expand $\sin(\phi) \approx \phi$, which makes the equation linear again (and equivalent to Eq. (4.59)).
A more complicated non-linear problem we had seen was the two-body or Kepler problem, e.g. Eq. (2.67) with inverse square law

$$\mu \ddot{r} = -\frac{GmM}{r^2} + \frac{\ell^2}{\mu r^3}. \tag{4.61}$$

The classification of further examples is e.g. the block sliding on sliding plane example 12 (linear), the bead on hoop, example 13 (nonlinear), heavy-top Eq. (3.54) (nonlinear).

- From the examples above, one might think that linear and non-linear systems are about equally common. In fact, in reality the large majority of systems are non-linear. The reason linear systems are nonetheless the more common ones considered in lectures and books, is that near equilibrium one can very often linearise the description, as shown above and for the general case in section 3.6.2. One could say that despite most systems being non-linear, luckily[12], we can frequently look at linear problems nonetheless, thanks to linearisation.
- The most important difference in the solution and math between the two systems, is that the superposition principle does not hold for non-linear systems. In a linear system, if you found two solutions $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$ of the dynamics, their sum is also a solution. Further, since the equations of motion are typically second order in time, once we have two such solutions, with arbitrary coefficients we can then write the general solution. Since it is only based on two ingredients, the complexity of such a solution is greatly limited. None of this holds for a non-linear equation, where even a single equation can have a much larger set of qualitatively different solutions.

4.7.2 The driven simple pendulum

In the list above, one of the simplest non-linear systems we have looked at is the simple pendulum. However we have seen in example 41, that despite the non-linearity the pendulum has a nicely sorted phase-space and regular behaviour, either showing oscillations or librations. This is a consequence of energy conservation, or the fact that the system is integrable (see section 4.4.2).

What if we now break integrability? We would for example expect energy to cease being conserved, if we externally drive the system. Since we will need it later, let us already introduce the

Driven damped pendulum (DDP) with equation of motion

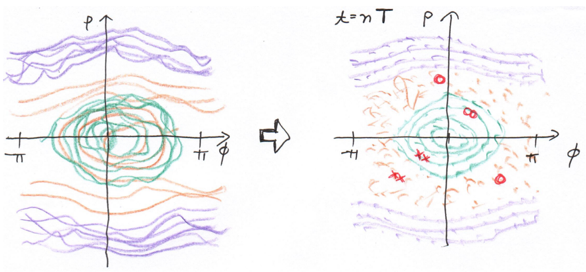
$$\ddot{\phi} + 2\gamma\dot{\phi} + \omega_0^2 \sin\phi = \kappa\omega_0^2 \cos(\omega t) \tag{4.62}$$

[12] Since a linear problem is typically easier

- Please look at the TT book for an extensive derivation of the ingredients. I use the same notation as TT here. If you look at example 41, you can reach the main part of (4.62) by differentiation the $\dot{\phi}$ equation wrt. time and then inserting $\dot{p}$, finding $\omega_0 = \sqrt{g/\ell}$. For the two new terms:
- Damping due to the term $2\gamma\dot{\phi}$ is added ad-hoc without derivation, we set γ to zero for now and shall only use it in section 4.7.4. We then have just a driven pendulum (DP).
- The driving force $\kappa\omega_0^2 \cos(\omega t)$ could be added on the level of the Lagrangian or Hamiltonian via a time dependent addition to the potential energy $V(t) = \kappa\omega_0^2 \cos(\omega t)\phi$.

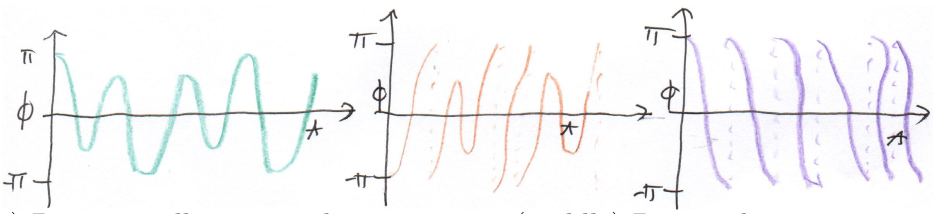
Let us now look at what consequences driving has for the phase-space trajectories:

Example 49, Stroboscopic Poincaré section: The first thing that we loose when driving the system ($\kappa \neq 0$), is the nice features that phase space trajectories cannot cross. They now can cross (at different times), as shown in the figure below. This makes the phase space diagram much less clear than before. The problem can be addressed, by moving to a Stroboscopic Poincaré plot of phase-space. For that, instead of following the phase-space trajectory at all times drawing $[q(t), p(t)]$, we only make a "dot" after an integer completion of driving periods $[q(nT), p(nT)]$, where $n \in \mathbb{Z}$, and $T = 2\pi/\omega$, where ω is the driving frequency in Eq. (4.62). It turns out this quite tidies up the picture, where we now can see regular structures as before (green, violet) and an irregular region (orange).



left: (left) Drawing of a collection of phase-space trajectories as in example 41 (samish color coding), but now for a driven pendulum. (right) Stroboscopic Poincaré section as explained in the text, for the same setup.

To understand more how a figure like the above comes about, let us look at one characteristic trajectory per colored region.



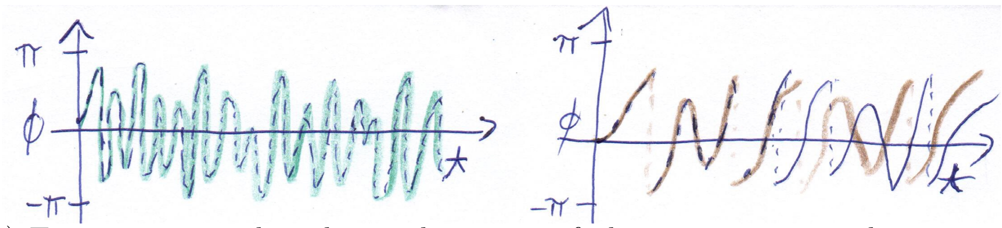
top: (left) Driven oscillatory regular trajectory. (middle) Driven chaotic trajectory. (right) Driven regular libration.

Through the stroboscopic recording we just pick certain snapshots of these trajectories, but if the underlying motion is still constrained on a regular surface in phase-space (as is the case), taking more and more snapshots still traces these regular structures.
Of most interest are the irregular trajectories (orange), that cover a large area of phase-space.

- To explain the name "Poincaré section": The main use of the concept is not (only) for time-dependent 1D systems, like here, but instead to visualize phase-spaces for higher-dimensional system. Since we cannot draw in more than 2 dimensions, what one does then is take a 2D cut or "section" of phase-space, and draw a dot for every trajectory that crosses that cut. For example in a 2D non-linear oscillator with phase space variables q_x, q_y, p_x, p_y, we can take the cut at $q_y = p_y = 0$ and draw a dot whenever a trajectory crosses these values.
- The irregular trajectories in the example above are an example for what is called "chaotic dynamics", which we now look at in more detail nextly.

4.7.3 Chaos

We had see in example 49 that some trajectories of the driven simple pendulum behave rather erratically, ending up as time goes on in lots of different corners of phase space. In particular, even though we may have started off two trajectories with nearly equal initial conditions (red crosses in the right panel in orange region), if we started them in the messy region of phase space, they might end up in quite different places (red circles in orange region). We show these again in a representations as function of time below. This does not happen for trajectories in the regular regions (same style symbols in green region).



top: (left) Trajectory started in the regular region of phase-space compared second trajectory with initial conditions differing by factors 0.001 overlaid as dashed line. (right) The same concept for two trajectories started in the chaotic region of phase-space.

We can characterise a systems sensitivity to the initial conditions, by the evolution of the separation between trajectories

$$d(t) = |\mathbf{q}_1(t) - \mathbf{q}_2(t)|. \tag{4.63}$$

Consider two trajectories that begin from very similar initial conditions, such that $d(t) \approx 0$ (but not fully zero).

If $d(t)$ then remains small at all times, the system is not sensitive to initial conditions. However if $d(t)$ can increase by orders of magnitude during time evolution it is sensitive. Particularly we shall see that there are systems where d becomes as large as the entire phase-space, even if it starts of infinitesimal.

We can quantify this using the

Lyapunov exponent The Lyapunov exponent of a dynamical system is λ, if

$$d(t) \sim e^{\lambda t}, \tag{4.64}$$

on average and for large times.

- The precise definition is somewhat more complicated, this is just to give the gist of the idea.
- Lyapunov exponents are used to make sure if a system is chaotic ($\lambda > 0$) or not ($\lambda < 0$), and if it is chaotic to quantify "how chaotic" it is.

With this we can now quite precisely define what we mean by

Chaos A dynamical system is called chaotic, if motion is bounded but nonetheless displays sensitivity to initial conditions as discussed above, or equivalently has a positive largest Lyapunov exponent.

- A linear finite dimensional system such as Eq. (3.60) cannot be chaotic. *Proof, exercise. Use that all solutions are either oscillatory (so not sensitive to initial conditions) or exponential, and use the superposition principle.* There are some examples of infinite dimensional linear systems, namely wave-equations, that show behaviour associated with chaos. Nonetheless, chaos is typically associated with non-linear mechanics.
- Integrable systems cannot be chaotic, because we can find a canonical transformation to describe them with action-angle variables as in example 43. The evolution equations in these new variables are then necessarily trivial and linear.
- Even though, classical mechanics allows in principle to always determine the future of a mechanical system without uncertainty, we realize now that this is in practice not possible for chaotic systems: Since they are sensitive to the tiniest variation of the initial conditions, and it is typically impossible to know the initial conditions to arbitrary precision, also classical chaotic systems become indeterministic (as quantum systems always are).

An unsatisfactory aspect of the discussion so far, that was not solved until about 1960, is the following: We had seen that the undriven pendulum is integrable with hence regular dynamics, while the driving breaks integrability and may lead to chaos. However, what if the driving is very weak, intuitively, we would expect that this cannot change the dynamics (and hence e.g. the phase space) very dramatically, while the transitions from an integrable to a chaotic systems seems rather abrupt. Reassuringly it is however continuous, which is expressed in the

Kolmogorov-Arnold-Moser (KAM) theorem
Let $\mathcal{H}_0$ be an integrable Hamiltonian, and $\Delta\mathcal{H}$ a small perturbations that breaks integrability. As long as the perturbation is weak[a], there will still be regular regions (KAM torii) in phase space.

[a]Other terms and conditions apply.

- We could see this in example 49 already. The regular regions for oscillations and librations are the "surviving" KAM torii. Only near the earlier separatrix, these get dissolved into chaotic regions.

4.7.4 Attractors and bifurcations

Let us now add dissipation to the DDP in Eq. (4.62), moving to $\gamma \neq 0$, and then looking at the concepts of attractors in phase-space and bifurcations. Before looking at the damped DDP, we review the damped and driven simple-harmonic-oscillator (SHO). We recover it from the DDP in the small angle approximation, where we can write $\sin\phi \approx \phi$ in Eq. (4.62) and reach

$$\ddot{\phi} + 2\gamma\dot{\phi} + \omega_0^2\phi = \kappa\omega_0^2 \cos{(\omega t)}. \tag{4.65}$$

This is called an in-homogeneous ODE of second order in time, because the function on the rhs is non-zero and does not contain ϕ. Suppose we know the solution $\phi_0(t)$ of the corresponding homogeneous system, which is the damped but not driven Harmonic oscillator. We can then see directly, that the total solution can be a combination $\phi_{\text{sho}}(t) = \phi_0(t) + \phi_s(t)$, where $\phi_s(t)$ is a special solution of the inhomogeneous system. We can guess the latter as a sinusoidal function with a phase, for which we find [see e.g. PHY101 or PHY106[13]]

$$\begin{aligned}
\phi_s(t) &= A(\omega)\cos{[\omega t + \varphi(\omega)]},\\
A(\omega) &= \frac{\kappa\omega_0^2}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\gamma^2\omega^2}},\\
\varphi(\omega) &= \arctan\left(\frac{2\gamma\omega}{\omega^2-\omega_0^2}\right).
\end{aligned} \tag{4.66}$$

We also know the solution of the damped-only oscillator is (for $\gamma < \omega_0$).

$$\begin{aligned}
\phi_0(t) &= \exp{[-\gamma t]}\left[C_1\cos{(\omega' t)} + C_2 \sin{(\omega' t)}\right],\\
\omega' &= \sqrt{\omega_0^2 - \gamma^2}.
\end{aligned} \tag{4.67}$$

The constants C_1 and C_2 depend on the initial conditions. We however see, that for times $t \gg \tau$ with $\tau = 1/\gamma$, the solution will essentially not contain the part (4.67) any more. This part is hence called transient solution, where transient means "existing for a short time after start only, and then

[13]When comparing e.g. with the result I gave you there, note the input equation is different here, as are the phase definitions (use of sin/cos).

disappearing". Regardless of the initial condition, we thus know that after some time the solution of the SHO will have settled onto the steady form (4.66).

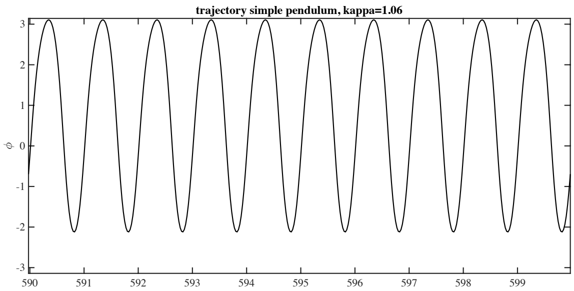
In the phase space of a driven dissipative system, this gives rise to the concept of an

Attractor This is a phase-space orbit onto which all trajectories started in some vicinity will evolve in the presence of dissipation and driving.

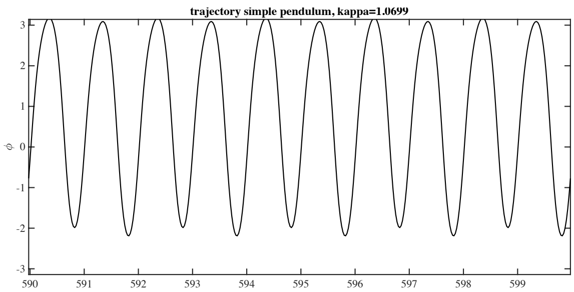
- The phase-space ellipse defined by (4.66) constitutes the attractor of the damped SHO. Trajectories in phase-space "spiral" onto this attractor (revisit assignment 1, Q4).

In a non-linear system, attractors can be more interesting and there can be more of them. First let's see what happens when we now move to the driven damped pendulum, instead of SHO.

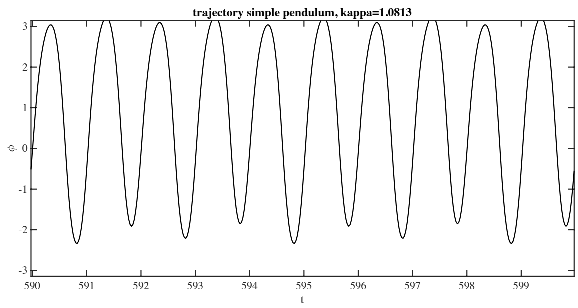
Example 50, Attractors in the DDP: We choose parameters $\omega = 2\pi$, $\omega_0 = 1.5\omega$, $\gamma = \omega_0/4$, and then solve Eq. (4.62) numerically for a few different drive strengths κ. As in the linear case, the motion has a transient that dies out on a time-scale given by γ, after which the solution settles into periodic motion.



left: Up to some $\kappa < \kappa_{\text{crit}}$, we see that the period of this motion is equal to the driving period.



left: For higher κ the period changes to twice the driving period, in what is called period doubling.

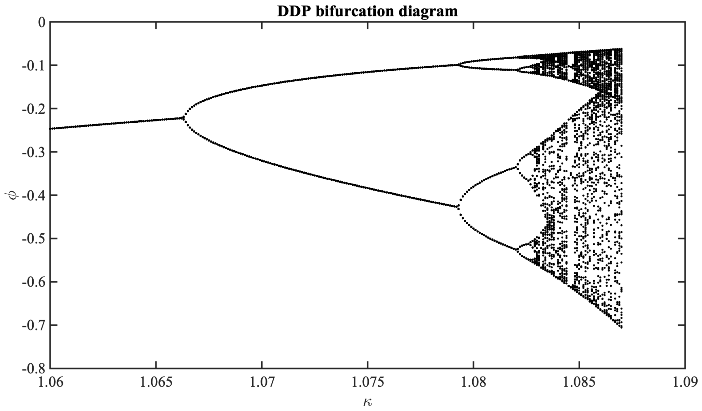


left: It then again doubles for even stronger driving, and now has a period four times the driving period.

- This is discussed in much more detail in TT, from where I took the parameters.

This qualitative change of the steady state solution happens fairly suddenly. To explore it in more detail, let us do a different kind of plot:

Example 51, Bifurcation diagram: We again vary κ as above, but with many more values. Instead of plotting trajectories as a function of time, we again do so stroboscopically.



left: We plot the angle $\phi(nT)$ after an integer $n \in \mathbb{N}$ number of periods $T = 2\pi/\omega$ of the drive, at late times where the transient has died out. Each thusly found ϕ then is the y-axis for a point, where κ is the x-axis. We plot many different n for the same κ.

On the left, the solution is periodic with the driving period. Thus, no matter what value of $\phi(nT)$ we accidentally hit, the next value $\phi((n+1)T)$ must be the same. Then we see a period doubling bifurcation, where for higher κ the solution has shifted to twice the drive period. Thus $\phi(nT)$ and $\phi((n+1)T)$ are now different, but then $\phi((n+2)T)$ must be again the same as $\phi(nT)$. We nextly see another period doubling bifurcation, followed by many more with increasing shorter κ intervals within. At the highest values of κ the system has become chaotic. Thus these sequential bifurcations are also called a route to chaos.

- If you zoom closely into the region of close-by successive bifurcation (which you can do in assignment 7), you notice what is called a fractal structure. No matter the zoom level, the tree-like structure of the plot will always look very similar.

5 Outlook

We now reached the end of this course. There is much still that we could not cover in the short time, such as

- D'Alembert's principle.
- Relativistic mechanics.
- Much more on nonlinear mechanics.
- Much more on canonical transformation, action angle variables and Hamilton-Jacobi theory.
- Canonical (classical) perturbation theory.
- Continuum mechanics: Shearing and strain within materials.

Some of this you can learn in follow up courses, when needed. All the topics above are covered or at least introduced among all the text-books given.

The End