LITTLEWOOD-PALEY THEORY

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Contents

1. Introduction

In this expository article we will study the L^p-inequalities for Littlewood-Paley operators. These operators consist of basic tools in analysis which allow us to decompose a function into pieces that have almost disjoint frequency supports.

2. Preliminaries and Notation

We will use the following standard notation :

- $C^\infty(\mathbb{R}^n)-$ space of infinitely differentiable functions on $\mathbb{R}^n$.
- $C_c^\infty(\mathbb{R}^n)-$ space of compactly supported $C^\infty(\mathbb{R}^n)-$functions.
- $\mathcal{S}(\mathbb{R}^n)-$ space of Schwartz class functions on $\mathbb{R}^n$
- $\mathcal{S}'(\mathbb{R}^n)-$ space of all tempered distributions on $\mathbb{R}^n$.
- $L^p(\mathbb{R}^n)-$ space of $p-$integrable functions on $\mathbb{R}^n$ for $0 < p < \infty$.
- $L^\infty(\mathbb{R}^n)-$ Banach space of essentially bounded measurable functions on $\mathbb{R}^n$.

Next, we define some basic operations on function spaces which will be used frequently.

Definition 2.1. *Let f, g be Lebesgue measurable functions on $\mathbb{R}^n$. Define,*

(1) *Translation:* $\tau_y f(x) := f(x-y),\ \ x, y \in \mathbb{R}^n$.
(2) *Dilation :* $D_\lambda f(x) := f(\lambda x),\ \ \lambda > 0, x \in \mathbb{R}^n$.

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(3) *Modulation:* $M_\xi f(x) := e^{2\pi i x.\xi} f(x),\ \xi, y \in \mathbb{R}^n.$
(4) *Convolution:* $f * g(x) := \int_{\mathbb{R}^n} f(y) g(x-y) dy,\ x \in \mathbb{R}^n,$ *and* $f, g \in L^1(\mathbb{R}^n).$

Definition 2.2 (Fourier transform)**.** *Let $f \in L^1(\mathbb{R}^n)$ then the Fourier transform of f is defined as the function*

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x.\xi} dx,\ \xi \in \mathbb{R}^n.$$

The Fourier transform satisfies the following relations with respect to operations of translation, dilation, modulation, and convolution.

Proposition 2.3. *Let $f, g \in L^1(\mathbb{R}^n)$. Then we have*

(1) $\widehat{\tau_y f}(\xi) = e^{-2\pi i \xi.y} \hat{f}(\xi) = M_{-y}\hat{f}(\xi), \quad \xi, y \in \mathbb{R}^n.$
(2) $\widehat{D_\lambda f}(\xi) = \frac{1}{\lambda^n}\hat{f}(\frac{\xi}{\lambda}) = \frac{1}{\lambda^n} D_{\frac{1}{\lambda}}\hat{f}(\xi),\ \ \lambda > 0, \xi \in \mathbb{R}^n.$
(3) $\widehat{M_\eta f}(\xi) = \hat{f}(\xi - \eta) = \tau_\eta \hat{f}(\xi),\ \ \xi, \eta \in \mathbb{R}^n.$
(4) $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi),\ \ \xi \in \mathbb{R}^n.$

Proof. The proof of Proposition 2.3 easily follows by doing a change of variables in the definition of Fourier transform. This may be left as an exercise. □

Theorem 2.4. *The Fourier transform satisfies the following L^p-estimates :*

(1) $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)},\ \ \forall f \in L^1(\mathbb{R}^n).$
(2) $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)},\ \forall f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *(Plancherel Theorem).*
(3) $\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)},\ \forall f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), 1 < p < 2$ *(Hausdorff-Young inequality), where p' denotes the conjugate index of p and is given by $\frac{1}{p} + \frac{1}{p'} = 1$.*

We shall use the notation $A \lesssim B$ if and only if $A \le CB$ for some constant $C > 0$ and $A \simeq B$ if and only if $A \lesssim B$ and $B \lesssim A$. In the later case we say that the two quantities are equivalent.

2.1. **Maximal functions.**

Definition 2.5 (Hardy-Littlewood maximal function)**.** *For a locally integrable functions f on $\mathbb{R}^n$, the classical Hardy-Littlewood maximal function $\mathcal{M}(f)(x)$ is defined by*

$$\mathcal{M}(f)(x) \ := \ \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1}$$

where the supremum is taken over all $n-$dimensional cubes Q containing x with sides parallel to coordinate axes and $|Q|$ is the measure of the cube Q.

Clearly, $\mathcal{M}$ is a sub-linear operator and it maps $L^\infty(\mathbb{R}^n)$ into itself, i.e.,

$$\|\mathcal{M}f\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}.$$

The L^p-boundedness properties of the Hardy-Littlewood maximal operator $\mathcal{M}$ are given in the following theorem :

Theorem 2.6. *For all functions $f \in \mathcal{S}(\mathbb{R}^n)$, there exists a constant $C > 0$ such that we have*

(1) *Strong type (p,p) inequality :* $\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},\ 1 < p \le \infty$
(2) *Weak type $(1,1)$ inequality :* $|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}| \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda},\ \ \forall \lambda > 0$

For $0 < q < \infty$, define

$$\mathcal{M}_q(f)(x) \quad := \quad (\mathcal{M}(|f|^q)(x))^{\frac{1}{q}}. \tag{2}$$

Exercise 2.7. *Using the result of Theorem 2.6, find values of p for which the operator $\mathcal{M}_q$ maps $L^p(\mathbb{R}^n)$ into itself.*

Definition 2.8 (Sharp maximal function)**.** *For a locally integrable functions f on $\mathbb{R}^n$, the sharp maximal function $\mathcal{M}^\sharp(f)(x)$ is defined by*

$$\mathcal{M}^\sharp(f)(x) \quad := \quad \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy, \tag{3}$$

where f_Q stands for the average of f over Q, i.e., $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$.

In fact, we have the following equivalence :

$$\mathcal{M}^\sharp(f)(x) \quad \simeq \quad \sup_{x\in Q} \inf_{a\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(y) - a|dy. \tag{4}$$

It is clear that the right hand side of (4) is $\leq \mathcal{M}^\sharp(f)(x)$. For the opposite inequality observe that :

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(y) - f_Q|dy \quad &\leq \quad \frac{1}{|Q|}\int_Q |f(y) - a| + |f_Q - a|dy \\ &\leq \quad \frac{2}{|Q|}\int_Q |f(y) - a|dy.\end{aligned}$$

This proves the opposite inequality.

Further, note that from the definition of sharp maximal function, we have $\mathcal{M}^\sharp(f)(x) \leq 2\mathcal{M}(f)(x)$. Hence, the sharp maximal operator shares the same L^p boundedness properties as the classical Hardy-Littlewood maximal operator.

2.2. **Fourier multipliers.**

Theorem 2.9. *Let $\mathcal{T}$ be a bounded linear operator on $L^2(\mathbb{R}^n)$, and let K be a function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ such that if $f \in L^2(\mathbb{R}^n)$ has compact support then*

$$\mathcal{T}(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \;\; x \notin supp(f).$$

Further, suppose that K also satisfies

$$\int_{|x-y|>2|y-z|} |K(x,y) - K(x,z)|dx \quad \lesssim \quad 1, \tag{5}$$

$$\int_{|x-y|>2|x-z|} |K(x,y) - K(z,y)|dy \quad \lesssim \quad 1. \tag{6}$$

Then $\mathcal{T}$ is of weak type $(1,1)$ and strong type (p,p) for all $p,\; 1 < p < \infty$.

The kernel K which satisfies conditions of the above theorem, is said to be a Calderón-Zygmund kernel and the associated operator $\mathcal{T}$ is called a Calderón-Zygmund operator. If $K(x,y)$ is the form $K_1(x-y)$, for some K_1, the operator $\mathcal{T}$ becomes a convolution operator $f \mapsto K_1 * f$ and in this case the Calderón-Zygmund theorem can be restated as follows :

Theorem 2.10. *Let K be a tempered distribution in $\mathbb{R}^n$, which coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ and satisfies*

$$|\hat{K}(\xi)| \lesssim 1, \tag{7}$$

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dx \lesssim 1, \; y \in \mathbb{R}^n. \tag{8}$$

Then the operator $\mathcal{T}f = K * f$ is of weak type $(1,1)$ and is bounded on $L^p(\mathbb{R}^n)$ for all $p, \; 1 < p < \infty$.

The condition (8) is referred to as Hörmander condition.

Exercise 2.11. *Show that the Hörmander condition (8) holds if for every $x \neq 0$, we have*

$$|\nabla K(x)| \lesssim \frac{1}{|x|^{n+1}}. \tag{9}$$

In order to avoid notational difficulties we work in one dimensional setting.

3. Littlewood-Paley square functions on $\mathbb{R}$

Recall the classical Hilbert transform : For $f \in \mathcal{S}(\mathbb{R})$, the Hilbert transform is the singular integral operator given by :

$$H(f)(x) = p.v. \int_{\mathbb{R}} f(x-y)\frac{dy}{y}. \tag{10}$$

Or equivalently, in terms of Fourier transform

$$\widehat{H(f)}(\xi) = -isgn(\xi)\hat{f}(\xi), \tag{11}$$

where

$$sgn(\xi) = \begin{cases} 1, & \xi > 0 \\ 0, & \xi = 0 \\ -1, & \xi < 0. \end{cases}$$

The Hilbert transform is the prototype of Calderón-Zygmund operators. We know that H is of weak type $(1,1)$ and strong type (p,p) for all $p, \; 1 < p < \infty$.

Let $I = [a,b]$ be an interval in $\mathbb{R}$. Consider the linear operator given by

$$\widehat{S_I f}(\xi) = \chi_I(\xi)\hat{f}(\xi), \; f \in \mathcal{S}(\mathbb{R}).$$

It is easy to show that the operator S_I has the following relation with the Hilbert transform :

$$S_I = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b}).$$

Again, it is easy to prove that L^p boundedness properties of the Hilbert transform is equivalent to the ones of the operator S_I. Moreover, the operator norm $\|S_I\|_{L^p \to L^p}$ is independent of the interval I. Verify!

Recall the Plancherel theorem : For all $f \in L^2(\mathbb{R})$, we have

$$\|f\|_2 = \|\hat{f}\|_2. \tag{12}$$

Thus for $f \in L^2$, we can completely recover the quantitative information about function f from its Fourier transform. But, this is not the case for functions in $L^p, p \neq 2$. Verify !

Let $\{I_n\}_{n\in\mathbb{Z}}$ be a given sequence of disjoint intervals in $\mathbb{R}$ such that it forms a partition for $\mathbb{R}$. Then, using Plancherel theorem, we can rewrite equation (12) as follows :

$$\|f\|_2 = \left\| \left(\sum_{n\in\mathbb{Z}} |S_{I_n}(f)|^2 \right)^{\frac{1}{2}} \right\|_2 . \tag{13}$$

The square function on RHS in the above equation is referred as Littlewood-Paley square function. This formulation in terms of Littlewood-Paley square functions provides a partial substitute in L^p, $p \neq 2$, for results obtained from the Plancherel theorem. To be more precise we define Littlewood-Paley square functions as follows :

Definition 3.1. *Let $\{I_n\}_{n\in\mathbb{Z}}$ be a sequence of disjoint intervals in $\mathbb{R}$. For $f \in \mathcal{S}(\mathbb{R})$, the Littlewood-Paley square function associated with the sequence $\{I_n\}_{n\in\mathbb{Z}}$ is defined as*

$$Sf(x) := \left(\sum_{n\in\mathbb{Z}} |S_{I_n}(f)|^2 \right)^{\frac{1}{2}} . \tag{14}$$

We will prove L^p boundedness properties of Littlewood-Paley square functions associated with sequence of disjoint intervals, but before proceeding further we present some important results about vector valued extension for bounded linear operators.

3.1. **Vector valued extension for bounded linear operators.** We first present a theorem due to Marcinkiewicz and Zygmund which asserts l_2- extension of bounded linear operators. This result will play an important role in the proof of our Littlewood-Paley results. Moreover, it is interesting in its own right.

Theorem 3.2. *Let T be a bounded linear operator from $L^p(\mathbb{R})$ into itself. Then T admits an l_2-valued extension with norm bounded by a constant multiple of $\|T\|_{L^p\to L^p}$, i.e., we have*

$$\left\| \left(\sum_{n\in\mathbb{Z}} |Tf_n|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \|T\| \left\| \left(\sum_{n\in\mathbb{Z}} |f_n|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Proof. The proof of this theorem is an easy application of Khintchine's inequality (see Appendix for details).

I leave it as an exercise. Use Khintchine's inequality for the sequence $\{T(f_n)(x)\}$, to linearize the square term. Then use linearity and boundedness of the operator T to complete the proof. $\square$

Corollary 3.3. *Let $\{I_n\}_{n\in\mathbb{Z}}$ be an arbitrary sequence of intervals in $\mathbb{R}$. Then for all p, $1 < p < \infty$, we have*

$$\left\| \left(\sum_{n\in\mathbb{Z}} |S_{I_n}f_n|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \|T\| \left\| \left(\sum_{n\in\mathbb{Z}} |f_n|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{15}$$

Proof. Let I_n be of the form $[A_n, B_n]$. Then we know that

$$S_{I_n} = \frac{i}{2}(M_{A_n}HM_{-A_n} - M_{B_n}HM_{-B_n}).$$

Use this relation together with the result of previous theorem to complete the proof. $\square$

Remark 3.4. *Theorem 3.2 is also valid for bounded linear operators from L^p into L^q.*

Remark 3.5. *The result of Corollary 3.3 is also valid for l_r-valued extension for $r \neq 2$, i.e., for all p and r, $1 < p, r < \infty$, we have*

$$\left\| \left(\sum_{n \in \mathbb{Z}} |S_{I_n} f_n|^r \right)^{\frac{1}{r}} \right\|_p \lesssim \|T\| \left\| \left(\sum_{n \in \mathbb{Z}} |f_n|^r \right)^{\frac{1}{r}} \right\|_p , \tag{16}$$

where S_{I_n} is same as in Corollary 3.3. The proof of this is a consequence of vector valued Calderón-Zygmund theorem.

4. Dyadic Littlewood-Paley theorem

Theorem 4.1. *Let $1 < p < \infty$ and $\Delta_n = (-2^{n+1}, -2^n] \cup [2^n, 2^{n+1})$, $n \in \mathbb{Z}$. Then for all $f \in \mathcal{S}(\mathbb{R})$, we have*

$$\left\| \left(\sum_{n \in \mathbb{Z}} |S_{\Delta_n}(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \simeq \|f\|_{L^p(\mathbb{R})}. \tag{17}$$

Proof. We first observe that in (17), the left hand side inequality can be deduced using the right hand side inequality. Assume for a moment that we have proved the right hand side inequality in (17). For nice functions f, g consider,

$$\begin{aligned} \langle f, g \rangle &= \int_{\mathbb{R}} f(x)\overline{g(x)}dx \\ &= \int_{\mathbb{R}} \hat{f}(\xi)\hat{\overline{g}}(\xi)d\xi \\ &= \sum_{n \in \mathbb{Z}} \int_{I_n} \hat{f}(\xi)\hat{\overline{g}}(\xi)d\xi \\ &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} S_{\Delta_n} f(x) S_{\Delta_n} \overline{g}(x)dx. \end{aligned}$$

Hence,

$$\begin{aligned} |\langle f, g \rangle| &\leq \int_{\mathbb{R}} \left(\sum_{n \in \mathbb{Z}} |S_{\Delta_n} f(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} |S_{\Delta_n} \overline{g}(x)|^2 \right)^{\frac{1}{2}} dx \\ &\leq \left\| \left(\sum_{n \in \mathbb{Z}} |S_{\Delta_n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \left\| \left(\sum_{n \in \mathbb{Z}} |S_{\Delta_n} \overline{g}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\mathbb{R})} \\ &\lesssim \left\| \left(\sum_{n \in \mathbb{Z}} |S_{\Delta_n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \|g\|_{L^{p'}(\mathbb{R})}. \end{aligned}$$

This proves the left hand side inequality in (17).
Now we proceed to prove the right hand side inequality in (17). The proof is in two steps.
Step I : In this step, we consider an appropriate smooth square function and then prove its L^p boundedness.

Let $\psi \in \mathcal{S}(\mathbb{R})$ ne a non-negative function such that *supp* $\psi \subseteq [-4, -\frac{1}{2}] \cup [\frac{1}{2}, 4]$ and $\psi \equiv 1$ on $[-2, -1] \cup [1, 2]$. For $n \in \mathbb{Z}$, define $\psi_n(\xi) = \psi(2^{-n}\xi)$. Let T_n be the multiplier operator associated with symbol ψ_n, i.e., $\widehat{T_n(f)} = \psi_n \hat{f}$.

We claim that that for all $p,\ 1<p<\infty$ we have

$$\left\|\left(\sum_{n\in\mathbb{Z}}|T_n(f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R})}\lesssim\|f\|_{L^p(\mathbb{R})}. \tag{18}$$

When $p=2$, inequality (18) is an easy consequence of Plancherel Theorem. For

$$\begin{aligned}\left\|\left(\sum_{n\in\mathbb{Z}}|T_n(f)|^2\right)^{\frac{1}{2}}\right\|^2_{L^2(\mathbb{R})} &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}|T_n(f)(x)|^2dx\\ &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}|\widehat{T_n(f)}(\xi)|^2d\xi\\ &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}|\psi_n(\xi)\hat{f}(\xi)|^2d\xi\\ &\lesssim \|f\|_2^2.\end{aligned}$$

Here in the last inequality we have used that for each fixed $\xi\in\mathbb{R}$, at most 3 of the ψ_n are non-zero and hence $\sum\limits_{n\in\mathbb{Z}}|\psi_n(\xi)|^2\le C$.

We think of the inequality (18) as an l_2-valued inequality for the operator $T:f\to\{T_n(f)\}$ from $L^p(\mathbb{R})$ into $L^p(l_2)$. We claim that this operator T is an l_2-valued Calderón-Zygmund operator. So we need to show the following

(1) T is bounded from L^2 into $L^2(l_2)$, which we have already seen.
(2) The kernel of T, which is given by $\{\check{\psi}_n\}$, satisfies the Hörmander condition in l_2 sense.

Set $\Psi_n=\check{\psi}_n,\ n\in\mathbb{Z}$. From our previous discussion we know that in order to prove the Hörmander condition, it suffices to prove the following :

$$\|\{\Psi_n'(x)\}\|_{l_2}=(\sum_{n\in\mathbb{Z}}|\Psi_n'(x)|^2)^{\frac{1}{2}}\le C|x|^{-2}. \tag{19}$$

Since $\Psi\in\mathcal{S}(\mathbb{R})$, we have $|\Psi'(x)|\le C min\{1,|x|^{-3}\}$. Let $x\in\mathbb{R}$ be fixed. Choose $i\in\mathbb{Z}$ so that $2^{-i}\le|x|<2^{-i+1}$. Then we have

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}|\Psi_n'(x)|^2\right)^{\frac{1}{2}} &\le \sum_{n\in\mathbb{Z}}|\Psi_n'(x)|\\ &= \sum_{n\in\mathbb{Z}}2^{2n}|\Psi'(2^nx)|\\ &\le \sum_{n\le i-1}2^{2n}|\Psi'(2^nx)|+\sum_{n\ge i}2^{2n}|\Psi'(2^nx)|.\end{aligned}$$

Note that $2^{n-i}\le|2^nx|<2^{n-i+1}$ as $2^{-i}\le|x|<2^{-i+1}$. Hence for $n\le i-1,\ min\{1,|2^nx|^{-3}\}=1$ and for $n\ge i,\ min\{1,|2^nx|^{-3}\}=|2^nx|^{-3}$. Thus we have

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}|\Psi_n'(x)|^2\right)^{\frac{1}{2}} &\le C\left[\sum_{n\le i-1}2^{2n}+|x|^{-3}\sum_{n\ge i}2^{-n}\right]\\ &\le C|x|^{-2}.\end{aligned}$$

This shows that T is an l_2-valued Calderón-Zygmund operator. Hence applying l_2-valued Calderón-Zygmund theorem we conclude that for all $p,\ 1<p<\infty$ the operator T maps $L^p(\mathbb{R})$ into $L^p(l_2)$. This completes the proof of estimate (18).

Step II : Observe that $S_{\Delta_n}T_nf = S_{I_n}f$ as $\psi\equiv 1$ on Δ_n. Now, invoke l_2- valued extension result from Theorem 3.2 together with the estimate (18) to conclude that :

$$\begin{aligned}\left\|\left(\sum_{n\in\mathbb{Z}}|S_{\Delta_n}(f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R})} &= \left\|\left(\sum_{n\in\mathbb{Z}}|S_{\Delta_n}T_n(f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R})}\\ &\leq C\left\|\left(\sum_{n\in\mathbb{Z}}|T_n(f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R})}\\ &\lesssim \|f\|_{L^p(\mathbb{R})}.\end{aligned}$$

This finishes the proof of Theorem 4.1. $\square$

Exercise 4.2. *Let $\{\epsilon_n\}_{n\in\mathbb{Z}}$ be a random sequence of ±1. Consider $m(\xi)=\sum\limits_{n\in\mathbb{Z}}\epsilon_n\psi_n$, where ψ_n is the same as in the proof of previous theorem. Set $K=\check{m}$. Prove that*

(1) $\sup_{\xi\in\mathbb{R}}|m(\xi)|\lesssim 1$,
(2) $|K(x)|\lesssim\frac{1}{|x|}$,
(3) $|K'(x)|\lesssim\frac{1}{|x|^2}$.

Exercise 4.3. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a lacunary sequence of positive real numbers, i.e., there exists a number $\delta>1$ such that $\frac{\lambda_{n+1}}{\lambda_n}\geq\delta$ for all $n\in\mathbb{Z}$. Consider $\Delta_n=[-\lambda_{n+1},-\lambda_n]\cup[\lambda_n,\lambda_{n+1}]$. Imitating the proof of Theorem 4.1, prove the analogue of Theorem 4.1 for this sequence of intervals.*

Exercise 4.4. *Let $\{\epsilon_n\}$ be a random sequence of ±1. Consider $m(\xi)=\sum\limits_{n\in\mathbb{Z}}\epsilon_n\chi_{\Delta_n}$, where Δ_n is a lacunary sequence of intervals as defined earlier. Prove that m is an L^p-multiplier for all $p,\ 1<p<\infty$.*

Exercise 4.5. *Let $\mathcal{D}$ denote the collection of dyadic rectangles in $\mathbb{R}^2$, i.e., $\mathcal{D}=\{I\times I'$: I and I' are dyadic intervals$\}$. Consider $\widehat{S^i_nf}(\xi_1,\xi_2)=\chi_{[2^n,2^{n+1}]}(\xi_i)\hat{f}(\xi_1,\xi_2),\ i=1,2,n\in\mathbb{Z}$. Prove that for all $p,\ 1<p<\infty$, we have*

$$\left\|\left(\sum_{n,m\in\mathbb{Z}}|S^1_nS^2_m(f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^2)} \simeq \|f\|_{L^p(\mathbb{R}^2)}.$$

5. Carleson's Littlewood-Paley theorem

In 1967, Carleson considered the Littlewood-Paley square function associated with the sequence $I_n=[n,n+1],\ n\in\mathbb{Z}$, and proved that :

Theorem 5.1. *Let $2\leq p<\infty$. Then for all $f\in\mathcal{S}(\mathbb{R})$, we have*

$$\left\|\left(\sum_{n\in\mathbb{Z}}|S_{[n,n+1]}(f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}. \tag{20}$$

Proof. When $p = 2$, this is an easy consequence of Plancherel theorem.

Our strategy is essentially the same as earlier. We will find an appropriate smooth square function and then prove its boundedness.

Let $\phi \in \mathcal{S}(\mathbb{R})$ be such that *supp* $\phi \subseteq [-\frac{1}{4}, \frac{5}{4}]$ and $\phi \equiv 1$ on $[0,1]$. For $n \in \mathbb{Z}$, define $\phi_n(\xi) = \phi(\xi - n)$. Note that *supp* $\phi_n \subseteq [n - \frac{1}{4}, n + \frac{5}{4}]$ and $\phi_n \equiv 1$ on $[n, n+1]$. Let T_n be the multiplier operator associated with ϕ_n, i.e., $\widehat{T_n(f)} = \phi_n \hat{f}$. Consider the smooth square function :

$$T(f)(x) = \left(\sum_{n \in \mathbb{Z}} |T_n(f)|^2 \right)^{\frac{1}{2}}.$$

Since $S_{[n,n+1]}(f) = S_{[n,n+1]}T_n(f)$, square function $T(f)$ is an appropriate one, i.e., in order to prove inequality (5.1), it suffices to prove that for $p > 2$, we have

$$\|T(f)\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}. \tag{21}$$

In order to prove inequality (21), it is enough to prove that for almost every $x \in \mathbb{R}$, $T(f)(x)$ satisfies the following pointwise estimate :

$$T(f)(x) \ \leq \ C \left(\mathcal{M}(|f|^2)(x)\right)^{\frac{1}{2}} = C\mathcal{M}_2(f)(x) \tag{22}$$

where $\mathcal{M}$ is the Hardy- Littlewood maximal operator and C is a constant independent of f. This estimate would give us the desired result using boundedness of the operator $\mathcal{M}_2$.

Let $\mathbf{a} = \{a_n\}_{n \in \mathbb{Z}} \in l_2(\mathbb{Z})$ be such that $\|\mathbf{a}\|_{l_2} = 1$. It suffices to prove that for a.e. $x \in \mathbb{R}$, we have the following :

$$\left| \sum_n a_n T_n(f)(x) \right| \ \lesssim \ \mathcal{M}_2(f)(x),$$

where C is independent of $\mathbf{a}$. Consider

$$\begin{aligned} \sum_n a_n T_n(f)(x) &= \sum_n a_n \int_{\mathbb{R}} f(x-y)\check{\phi}_n(y)dy \\ &= \int_{\mathbb{R}} f(x-y)\check{\phi}(y) \sum_n a_n e^{2\pi i n y} dy \\ &= \int_{\mathbb{R}} f(x-y)\check{\phi}(y)h(y)dy, \end{aligned}$$

where h is a 1−periodic function given by its Fourier series $h(y) = \sum_n a_n e^{2\pi i n y}$. Moreover, for any $c \in \mathbb{R}$, we have $\int_c^{c+1} |h(y)|^2 dy = 1$. Hence

$$\begin{aligned} \sum_n a_n T_n(f)(x) &= \int_{\mathbb{R}} f(x-y)\check{\phi}(y)h(y)dy \\ &= \left(\int_I + \sum_{n=1}^{\infty} \int_{2^n I \setminus 2^{n-1} I} \right) f(x-y)\check{\phi}(y)h(y)dy \end{aligned}$$

Since $\phi \in \mathcal{S}(\mathbb{R})$, there exists a constant C_N such that $|\check{\phi}(y)| \leq \frac{C_N}{(1+|y|)^N}$ for all $N \in \mathbb{N}$. Using this decay property of ϕ and Hölder's inequality, we obtain

$$\begin{aligned}
\big|\sum_n a_n T_n(f)(x)\big| &\leq C_N \sum_{n=0}^{\infty} 2^{-nN} \int_{2^n I} |f(x-y)h(y)h(y)| dy \\
&\lesssim \sum_{n=0}^{\infty} 2^{-nN} 2^{n/2} \left(\int_{2^n I} |f(x-y)|^2 dy\right)^{1/2} \\
&= \sum_{n=0}^{\infty} 2^{-n(N-1)} \left(2^{-n} \int_{2^n I} |f(x-y)|^2 dy\right)^{1/2} \\
&\lesssim \sum_{n=0}^{\infty} 2^{-n(N-1)} \left(\mathcal{M}(|f|^2)(x)\right)^{\frac{1}{2}} \\
&\lesssim \mathcal{M}_2(f)(x)
\end{aligned}$$

This completes the proof of Theorem 5.1. □

Proposition 5.2. *$p \geq 2$ is a necessary condition in Theorem 5.1.*

Proof. For $N \in \mathbb{N}$ consider $\hat{f}_N = \chi_{[0,N]}$. Observe that for all $n, 0 \leq n \leq N-1$, we have

$$\begin{aligned}
S_{[n,n+1]}(f_N)(x) &= \int_{\mathbb{R}} \chi_{[0,N]} \chi_{[n,n+1]}(\xi) e^{2\pi i \xi x} d\xi \\
&= \int_{\mathbb{R}} \chi_{[n,n+1]}(\xi) e^{2\pi i \xi x} d\xi \\
&= e^{2\pi i n x} \int_{\mathbb{R}} \chi_{[0,1]}(\xi) e^{2\pi i \xi x} d\xi.
\end{aligned}$$

Hence,

$$|S_{[n,n+1]}(f_N)(x)| = |F(x)|, \ 0 \leq n \leq N-1,$$

where $F(x) = \int_{\mathbb{R}} \chi_{[0,1]}(\xi) e^{2\pi i \xi x} d\xi$, is a fixed function (independent of n and N). Moreover, $F \in L^p(\mathbb{R}), \ p > 1$. Then, for almost every $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left(\sum_{n \in \mathbb{Z}} |S_{[n,n+1]}(f_N)|^2\right)^{\frac{1}{2}} &\geq \left(\sum_{n=0}^{N-1} |S_{[n,n+1]}(f_N)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{n=0}^{N-1} |F(x)|^2\right)^{\frac{1}{2}} \\
&= N^{1/2} F(x).
\end{aligned}$$

Applying inequality (20) to function f_N, we get that

$$\begin{aligned}
N^{1/2} &\lesssim \left\| \left(\sum_{n \in \mathbb{Z}} |S_{[n,n+1]}(f_N)(x)|^2\right)^{1/2} \right\|_{L^p(\mathbb{R})} \\
&\lesssim \|f_N\|_{L^p(\mathbb{R})} \\
&\lesssim N^{1/p'},
\end{aligned}$$

where the implicit constant does not depend on N. Choosing N large enough we conclude that $p \geq 2$. □

Exercise 5.3. *We have used the Hardy-Littlewood maximal function to prove Theorem 5.1. Try to prove it without using the maximal function.*

Exercise 5.4. *Let $\{I_n\}_{n\in\mathbb{Z}}$ be a sequence of intervals in $\mathbb{R}$ such that $I_n \subseteq [n, n+1]$ for every $n \in \mathbb{Z}$. Prove that for all $p,\ 2 \le p < \infty$, we have*

$$\left\| \left(\sum_{n\in\mathbb{Z}} |S_{I_n}(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})},\ \forall f \in \mathcal{S}(\mathbb{R}).$$

Exercise 5.5. *Formulate the higher dimensional analogue of Theorem 5.1 and prove it.*

6. Rubio de Francia's Littlewood-Paley theorem for arbitrary intervals

In both the previous Littlewood-Paley results, sequence of intervals have very specific properties. In the first case, intervals are dilates of each other by a factor of 2, whereas in the second case they are integer translates of each other. Now we consider arbitrary sequences of disjoint intervals and prove boundedness of the associated Littlewood-Paley square function. This was proved in 1985 by Rubio de Francia [3].

Theorem 6.1. *Let $\{I_n\}_{n\in\mathbb{Z}}$ be an arbitrary sequence of disjoint intervals in $\mathbb{R}$. Then for all $p,\ 2 \le p < \infty$, we have*

$$\left\| \left(\sum_{n\in\mathbb{Z}} |S_{I_n}(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}\ \forall f \in \mathcal{S}(\mathbb{R}). \tag{23}$$

The proof of this theorem is quite involved. However, the mail idea of the is essentially the same as in previous Theorems 4.1 and 5.1, i.e., to say, we will first find a suitable smooth square function and then prove its boundedness. But, finding an appropriate smooth square function is not as straight forward as earlier. Also, it is hard to prove estimates for smooth square function in this general setting. We will be following the original paper of Rubio de Francia [3].

Proof. **Proof of Theorem 6.1 :** The proof is done in several steps.

6.1. Step I- Reduction to the case of well-distributed collection.

We first regularize the given collection of intervals. This step is very crucial and allows us to define an appropriate smooth square function.

Well-distributed collection of intervals : A collection of intervals $\{I\}_{I\in\mathcal{I}}$ is said to be well-distributed if there exists a constant $C > 0$ such that

$$\sum_{I\in\mathcal{I}} \chi_{2I}(x) \le C,\ \forall x \in \mathbb{R},$$

where $2I$ stands for the dilated interval having the same center as I and $|2I| = 2|I|$.

Consider the interval $I = (0,1)$ and define the Whitney decomposition of I as follows :

$$W(I) := \left\{ \left[\frac{2^{-(k+1)}}{3}, \frac{2^{-k}}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[1 - \frac{2^{-k}}{3}, 1 - \frac{2^{-(k+1)}}{3}\right] : k \in \mathbb{N} \cup \{0\} \right\}.$$

It is clear that intervals in $W(I)$ are disjoint and form a covering of I. Moreover, the collection $W(I)$ has the following properties :

(1) $2J \subseteq I$ for every $J \in W(I)$ and
(2) $\sum_{J\in W(I)} \chi_{2J}(x) \le 5,\ \forall x \in \mathbb{R}.$

Proof of (1) : Let $J \in W(I)$. Then J is either $[\frac{1}{3}, \frac{2}{3}]$ or is of the form $[\frac{2^{-(k+1)}}{3}, \frac{2^{-k}}{3}]$ or $[1-\frac{2^{-k}}{3}, 1-\frac{2^{-(k+1)}}{3}]$ for some $k \in \mathbb{N} \cup \{0\}$. If $J = [\frac{1}{3}, \frac{2}{3}]$, then it is obvious that $2J \subseteq I$. Assume that $J = [\frac{2^{-(k+1)}}{3}, \frac{2^{-k}}{3}]$ for some $k \in \mathbb{N} \cup \{0\}$. Then length of J, $|J| = \frac{2^{-(k+1)}}{3}$ and the center of J, $C(J) = 2^{-(k+2)}$. Hence

$$\begin{aligned}
2J &= [C(J)-|J|, C(J)+|J|] \\
&= \left[2^{-(k+2)} - \frac{2^{-(k+1)}}{3}, 2^{-(k+2)} + \frac{2^{-(k+1)}}{3}\right] \\
&= \left[\frac{2^{-(k+1)}}{6}, \frac{2^{-(k+1)}5}{6}\right] \\
&\subseteq [0,1], \ \forall k \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

The proof is similar when $J = [1-\frac{2^{-k}}{3}, 1-\frac{2^{-(k+1)}}{3}]$ for some $k \in \mathbb{N} \cup \{0\}$.

Proof of (2) : Note that if $J_1, J_2 \in W(I)$ are such that $2J_1 \cap 2J_2 \neq \phi$, then both the intervals necessarily sit either to right or to left of $[\frac{1}{3}, \frac{2}{3}]$.
Assume that $J_1 = [\frac{2^{-(k+1)}}{3}, \frac{2^{-k}}{3}]$ and $J_2 = [\frac{2^{-(l+1)}}{3}, \frac{2^{-l}}{3}]$ for some $k, l \in \mathbb{N} \cup \{0\}$ and $2J_1 \cap 2J_2 \neq \phi$. We know that $2J_1 = \left[\frac{2^{-(k+1)}}{6}, \frac{2^{-(k+1)}5}{6}\right]$ and $2J_2 = \left[\frac{2^{-(l+1)}}{6}, \frac{2^{-(l+1)}5}{6}\right]$. Without loss of generality we may assume that J_1 sits to the left of J_2. Then, we should have $k > l$. Since $2J_1 \cap 2J_2 \neq \phi$, we must have that $\frac{2^{-(l+1)}}{6} < \frac{2^{-(k+1)}5}{6}$, which in turn implies that $2^{k-l} < 10$.
Note that for each fixed l, we have only four choices of $k > l$, namely $k = l, l+1, l+2, l+3$, such that $2^{k-l} < 10$.

This proves that for any $J \in W(I)$, $2J$ can have non-empty intersection with at most four intervals in $\{2H \ : H \in W(I)\}$. This completes the proof of (2).

Observe that the definition of Whitney decomposition is invariant under translation and dilation. Hence, for an arbitrary interval I, we can define its Whitney decomposition $W(I)$ as previously. Moreover, the collection $W(I)$ will have previously mentioned properties (1) and (2).

Given an interval I, consider the operator :

$$S^I f(x) \ := \ \left(\sum_{J \in W(I)} |S_J(f)|^2\right)^{\frac{1}{2}}, \ f \in \mathcal{S}(\mathbb{R}). \tag{24}$$

We would like to remark that $S^I f$ is a a variant of dyadic Littlewood-Paley square function. We have the following lemma :

Lemma 6.2. *Let $\{I_n\}_{n\in\mathbb{Z}}$ be a sequence of disjoint intervals in $\mathbb{R}$. Then for all p, $1 < p < \infty$, we have*

$$\left\|\left(\sum_{n\in\mathbb{Z}} |S_{I_n}(f)|^2\right)^{\frac{1}{2}}\right\|_p \ \simeq \ \left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(f)|^2\right)^{\frac{1}{2}}\right\|_p \tag{25}$$

Proof. Since the operators S^{I_n} are uniformly bounded on $L^2(\omega)$ for all weights $\omega \in A_2$. Then for all p, $1 < p < \infty$, we have the following vector valued extension :

$$\left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(f_n)|^2\right)^{\frac{1}{2}}\right\|_p \ \lesssim \ \left\|\left(\sum_{n\in\mathbb{Z}} |f_n|^2\right)^{\frac{1}{2}}\right\|_p. \tag{26}$$

Set $f_n = S_{I_n}(f)$. Then by definition of $S^{I_n}f$ (see (24))

$$\begin{aligned} S^{I_n}f_n &= \left(\sum_{J\in W(I)} |S_J(f_n)|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{J\in W(I)} |S_J f|^2\right)^{\frac{1}{2}} \\ &= S^{I_n}f. \end{aligned}$$

Applying vector valued estimate (26) to this choice of functions, we get

$$\begin{aligned} \left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(f)|^2\right)^{\frac{1}{2}}\right\|_p &= \left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(S_{I_n}f)|^2\right)^{\frac{1}{2}}\right\|_p \\ &\lesssim \left\|\left(\sum_{n\in\mathbb{Z}} |S_{I_n}(f)|^2\right)^{\frac{1}{2}}\right\|_p \end{aligned}$$

This proves $\lesssim$ inequality in (25). We use duality arguments to prove the opposite side inequality in (25). Consider,

$$\begin{aligned} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} S_{I_n}(f)(x)h_n(x)dx &= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} S_{I_n}(f)(x)h_n(x)dx \\ &= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \sum_{J\in W(I_n)} S_J S_{I_n}(f)(x) S_J(h_n)(x)dx \\ &= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \sum_{J\in W(I_n)} S_J(f)(x) S_J h_n(x)dx. \end{aligned}$$

Applying Hólder's inequality twice and using estimate (26), we get

$$\begin{aligned} \left|\int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} S_{I_n}(f)(x)h_n(x)dx\right| &\leq \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} \sum_{J\in W(I_n)} |S_J(f)(x)|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \sum_{J\in W(I_n)} |S_J(h_n)(x)|^2\right)^{\frac{1}{2}} dx \\ &= \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} |S^{I_n}(f)(x)|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} |S^{I_n}(h_n)(x)|^2\right)^{\frac{1}{2}} dx \\ &\leq \left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(f)|^2\right)^{\frac{1}{2}}\right\|_p \left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(h_n)|^2\right)^{\frac{1}{2}}\right\|_{p'} \\ &\lesssim \left\|\left(\sum_{n\in\mathbb{Z}} |S^{I_n}(f)|^2\right)^{\frac{1}{2}}\right\|_p \left\|\left(\sum_{n\in\mathbb{Z}} |h_n|^2\right)^{\frac{1}{2}}\right\|_{p'}. \end{aligned}$$

This estimate allows us to deduce $\gtrsim$ inequality in (25) using duality. ☐

We have proved that L^p norm of the square function associated with an arbitrary sequence of disjoint intervals $\{I_n\}_{n\in\mathbb{Z}}$ is equivalent to L^p norm of the square function associated with the collection of intervals $\mathcal{W} = \{J : J \in W(I_n) \textit{ for some } n \in \mathbb{Z}\}$. Moreover, we have proved that the collection $\mathcal{W}$ is a well-distributed collection of intervals. Thus it suffices to prove Theorem 6.1 with an additional assumption of well-distributiveness on the collection of intervals.

Now onwards we shall always assume that the collection of intervals under consideration is a well-distributed collection.

6.2. **Step II- Reduction to boundedness of smooth square function.** In this section we will find a suitable smooth square function. Then we will show that in order to prove Theorem 6.1, it suffices to prove analogous L^p estimates for this smooth square function.

We start with $\mathcal{W}-$ well-distributed collection of intervals. We further regularize this collection in the following sense : Divide each interval $I \in \mathcal{W}$ into seven consecutive intervals of equal lengths, i.e., $I = \cup_{i=1}^{7} I^i$ with $|I^i| = |I|/7$ for all $i = 1, 2, 3, ..., 7$.

For each $i = 1, 2, 3, ..., 7$, we consider the collection $\mathcal{W}_i = \{I^i \ : \ I \in \mathcal{W}\}$. We will prove Theorem 6.1 for each collection $\mathcal{W}_i$. Since all the seven collections $\mathcal{W}_i$ have same properties, it is enough to prove the result for one such collection. Let us fix one such collection of intervals and denote it by $\mathcal{I}$. Observe that we have

$$\sum_{I\in\mathcal{I}} \chi_{8I}(x) \leq 5, \ \forall x \in \mathbb{R}. \tag{27}$$

We label the intervals in $\mathcal{I}$ according to their sizes. For each integer $k \in \mathbb{Z}$, let $\mathcal{I}_k$ denote the collection $\{I_k^j\}_j = \{I \in \mathcal{I} \ : \ 2^k \leq |I| < 2^{k+1}\}$. With this new notation, we need to prove that for all $p, \ 2 \leq p < \infty$, we have

$$\left\| \left(\sum_{k,j} |S_{I_k^j}(f)(x)|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \|f\|_p. \tag{28}$$

For every k and j, choose n_k^j to be the first integer such that $n_k^j 2^k \in I_k^j$. This is possible because of the definition of I_k^j.

Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $\chi_{[-2,2]} \leq \hat{\psi} \leq \chi_{[-3,3]}$. Define

$$\psi_k^j(x) = 2^k \psi(2^k x) e^{2\pi i n_k^j 2^k x}.$$

Then the Fourier transform of ψ_k^j is given by

$$\hat{\psi}_k^j(\xi) = \hat{\psi}(2^{-k}\xi - n_k^j).$$

Usie the size condition $2^k \leq |I_k^j| < 2^{k+1}$ together with the choice of n_k^j, to show that $I_k^j \subseteq [2^k(n_k^j - 2), 2^k(n_k^j + 2)]$. Also, show that *supp* $\hat{\psi}_k^j \subseteq 8I_k^j$ and $\hat{\psi}_k^j \equiv 1$ on I_k^j.
We now consider the smooth square function associated with the sequence $\hat{\psi}_k^j$ defined as follows :

$$T(f)(x) := \left(\sum_{k,j} |T_k^j(f)(x)|^2 \right)^{\frac{1}{2}}, \tag{29}$$

where T_k^j is the convolution operator given by $T_k^j(f) = \psi_k^j * f$. Observe that $T(f)$ is the appropriate smooth square function as $S_{I_k^j}(f) = S_{I_k^j} T_k^j(f)$. Hence using vector valued arguments as earlier, we know that in order to prove inequality (28), it suffices to prove that for all $p, \ 2 \leq p < \infty$, we have

$$\|T(f)\|_p \ \lesssim \ \|f\|_p. \tag{30}$$

Next two sections are devoted to prove estimate (30) for the smooth square function.

6.3. **Step III- A key vector valued lemma.** This section is devoted to establsih a general result for vector valued operators and in the next section, we will show that the smooth square function under consideration falls into the setting of this result.

We consider the following setting : Let E be a Hilbert space and let $K(x,y)$ be an $E-$valued function defined on $\mathbb{R}^2$ such that $\|K(x,.)\|_E$ is locally integrable for each fixed $x \in \mathbb{R}$. We consider the following $E-$valued operator :

$$\mathcal{T}(f)(x) := \int_{\mathbb{R}} f(y)K(x,y)dy. \tag{31}$$

It is easy to check that $\mathcal{T}$ is well defined for functions $f \in C_c^\infty(\mathbb{R})$.

Given $x, z \in \mathbb{R}$ and $m \in \mathbb{Z}$, denote

$$I_m(x,z) = \{y \in \mathbb{R} \ : \ 2^m|x-z| < |y-z| \leq 2^{m+1}|x-z|.$$

Observe that $|I_m(x,z)| = 2^m|x-z|$.

Then, we have the following important result for the operator $\mathcal{T}$:

Lemma 6.3. *Let $\mathcal{T}$ be the operator given by (31). Suppose that $\mathcal{T}$ is bounded from $L^2(\mathbb{R})$ into $L^2(\mathbb{R},E)$. Further, assume that for some positive constants $A>0$ and $\alpha>1$, the kernel $K(x,y)$ satisfies*

$$\int_{I_m(x,z)} |\langle K(x,y)-K(z,y), \lambda\rangle_E|^2 dy \leq A^2 \frac{2^{-\alpha m}\|\lambda\|_E^2}{|x-z|}, \tag{32}$$

for every $x, z \in \mathbb{R}$, $\lambda \in E$, and $m \geq 1$.

Then, the $E-$valued operator $G(f)(x) := \|\mathcal{T}(f)(x)\|_E$ satisfies the following :

$$\mathcal{M}^\sharp(G(f))(x) \leq C(A,\alpha)\mathcal{M}_2(f)(x). \tag{33}$$

Proof. Given an $x \in \mathbb{R}$ and an interval I centered at x, define the vector $h_I \in E$ as

$$h_I = \int_{y \notin 2I} f(y)K(x,y)dy.$$

Consider,

$$\begin{aligned}
\mathcal{T}(f)(z) - h_I &= \int_{\mathbb{R}} f(y)K(z,y)dy - \int_{y\notin 2I} f(y)K(x,y)dy \\
&= \int_{y\in 2I} f(y)K(z,y)dy + \int_{y\notin 2I} f(y)(K(z,y)-K(x,y))dy. \\
&= \mathcal{T}(f_1)(z) + \int_{y\notin 2I} f(y)(K(z,y)-K(x,y))dy,
\end{aligned}$$

where $f_1 = f\chi_{2I}$. This implies that

$$\begin{aligned}
\frac{1}{|I|}\int_I \|\mathcal{T}(f)(z) - h_I\|_E dz &\leq \frac{1}{|I|}\int_I \|\mathcal{T}(f_1)(z)\|_E dz \\
&\quad + \frac{1}{|I|}\int_I \| \int_{y\notin 2I} f(y)(K(z,y)-K(x,y))dy\|_E dz \\
&= I_1 + I_2,
\end{aligned}$$

where I_1 and I_2 are first and second terms in RHS of the above expression.

We will estimate both these terms separately and show that each of them is dominated by $\mathcal{M}_2(f)$, which in turn would give us the desired result. Then,

$$\begin{aligned} I_1 &= \frac{1}{|I|}\int_I \|\mathcal{T}(f_1)(z)\|_E dz \\ &\leq \left(\frac{1}{|I|}\int_I \|\mathcal{T}(f_1)(z)\|_E^2 dz\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{|I|}\int_{\mathbb{R}} |f_1(z)|^2 dz\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{|I|}\int_{2I} |f(z)|^2 dz\right)^{\frac{1}{2}} \\ &\lesssim \mathcal{M}_2(f)(x). \end{aligned}$$

Now we proceed to estimate I_2. Let $g(z)$ denote $E-$valued function with $\|g(z)\|_E \leq 1$ for all $z \in I$. Then using duality we know that

$$I_2 = \sup_g \frac{1}{|I|}\int_I \left|\langle g(z), \int_{y\notin 2I} f(y)(K(z,y)-K(x,y))dy\rangle_E\right| dz,$$

where supremum is taken over all functions g defined as above. Let g be a fixed function described as above. Note that for all $x, z \in I$ and $y \notin 2I$, we have $\frac{|x-z|}{2} < |y-x|$. Then

$$\begin{aligned} & \frac{1}{|I|}\int_I \left|\langle g(z), \int_{y\notin 2I} f(y)(K(z,y)-K(x,y))dy\rangle_E\right| dz \\ \leq\ & \frac{1}{|I|}\int_I \int_{y\notin 2I} |f(y)| |\langle g(z), K(z,y)-K(x,y)\rangle_E| dy dz \\ \leq\ & \frac{1}{|I|}\int_I \sum_{m=1}^{\infty}\int_{y\in I_m(z,x)} |f(y)| |\langle g(z), K(z,y)-K(x,y)\rangle_E| dy dz \\ \leq\ & \frac{1}{|I|}\int_I \sum_{m=1}^{\infty}\left(\int_{I_m(z,x)} |f(y)|^2 dy\right)^{\frac{1}{2}}\left(\int_{I_m(z,x)} |\langle g(z), K(z,y)-K(x,y)\rangle_E|^2 dy\right)^{\frac{1}{2}} dz \\ \leq\ & \frac{1}{|I|}\int_I \sum_{m=1}^{\infty}\left(\int_{I_m(z,x)} |f(y)|^2 dy\right)^{\frac{1}{2}} \|g(z)\|_E \frac{A2^{-\alpha m/2}}{|x-z|^{1/2}} dz \\ =\ & \frac{1}{|I|}\int_I \sum_{m=1}^{\infty} A2^{-\alpha m/2}\left(\frac{2^m}{2^m|x-z|}\int_{I_m(z,x)} |f(y)|^2 dy\right)^{\frac{1}{2}} dz \\ \leq\ & C\frac{1}{|I|}\int_I \sum_{m=1}^{\infty} A2^{-\alpha m/2}2^{m/2}\left(\frac{1}{|I_m(z,x)|}\int_{I_m(z,x)} |f(y)|^2 dy\right)^{\frac{1}{2}} dz \\ \leq\ & C\mathcal{M}_2(f)(x) \end{aligned}$$

Here we have used that $\alpha > 1$ and hence the sum in the above is finite. This establishes inequality (33) and hence the proof of Lemma 6.3 is complete. □

6.4. **Step IV- Boundedness of the smooth square function.** The choice of ψ_k^j is such that for some constant $C > 0$, we have

$$\sum_{k,j} |\hat{\psi}_k^j(\xi)|^2 \leq C.$$

Hence by Plancherel theorem we get,

$$\|T(f)\|_2 \leq C\|f\|_2. \tag{34}$$

Our objective is to prove inequality (30) for all p, $2 < p < \infty$. In order to do this, we shall show that for almost every $x \in \mathbb{R}$, we have

$$\mathcal{M}^\sharp(T(f))(x) \lesssim \mathcal{M}_2(f)(x), \tag{35}$$

where $\mathcal{M}^\sharp(T(f))$ is the sharp maximal function of $T(f)$.

Once we have the above estimate, the desired result for square function may be obtained as follows :

$$\begin{aligned} \|T(f)\|_p &\lesssim \|\mathcal{M}^\sharp(T(f))\|_p \\ &\lesssim \|\mathcal{M}_2(f)\|_p \\ &\lesssim \|f\|_p,\ 2 < p \leq \infty. \end{aligned}$$

Thus we only need to establish the inequality (35). In order to prove this we will prove a general result for singular integral operators in vector valued setting. Now we need to prove that the square function under consideration falls into the setting of Lemma 6.3. Recall that square function is given as :

$$\begin{aligned} T(f)(x) &= \left(\sum_{k,j} |T_k^j(f)(x)|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k,j} |\psi_k^j * f(x)|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k,j} \left|\int_{\mathbb{R}} 2^k\psi(2^k(x-y))e^{-2\pi i n_k^j 2^k y} f(y)dy\right|^2\right)^{\frac{1}{2}}. \end{aligned}$$

We view the operator T as an l_2-valued operator and observe that its kernel is given by

$$K(x,y) = \{2^k\psi(2^k(x-y))e^{-2\pi i n_k^j 2^k y}\}_{k,j}. \tag{36}$$

We already know that T is bounded from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}, l_2)$. So, in order to apply the result of Lemma 6.3, we only need to establish estimate (32) for the kernel $K(x,y)$. Let $\lambda = \{\lambda_{k,j}\} \in l_2$ be such that $\|\lambda\|_2 = 1$. Then, we have

$$\begin{aligned} \langle K(x,y), \lambda\rangle &= \sum_{k,j} 2^k\psi(2^k(x-y))\overline{\lambda}_{k,j} e^{-2\pi i n_k^j 2^k y} \\ &= \sum_{k} 2^k\psi(2^k(x-y))h_k(2^k y), \end{aligned} \tag{37}$$

where for each k, h_k is a $1-$periodic function given by its Fourier series : $h_k(x) = \sum_j \overline{\lambda}_{k,j} e^{-2\pi i n_k^j x}$.

Note that the choice of integers n_k^j tells us that $n_k^j \neq n_k^{j'}$ if $j \neq j'$. This implies that

$$\int_a^{a+1} |h_k(x)|^2 dx \leq 1,\ \forall a \in \mathbb{R}, k \in \mathbb{Z}. \tag{38}$$

We would like to remark that we only need the above property of function h_k, in the proof. Recall that we are interested in proving the following :

$$\int_{I_m(x,z)} |\langle K(x,y)-K(z,y),\lambda\rangle|^2 dy \leq \frac{A^2 2^{-\alpha m}}{|x-z|}, \tag{39}$$

for some $A>0$ and $\alpha>1$.

It suffices to prove the above estimate assuming $z=0$ because this only amounts to translating h_k by a factor of $2^k z$ and we know that inequality (38) is invariant under translation. Thus our job is reduced to prove that

$$\int_{I_m(x,0)} |\langle K(x,y)-K(0,y),\lambda\rangle|^2 dy \leq \frac{A^2 2^{-\alpha m}}{|x|} \tag{40}$$

Further, observe that the above inequality does not change, if we replace x by $2x$, and thus, we can assume that $1 \leq |x| < 2$. Using estimate (37), we have

$$\begin{aligned}
&\left(\int_{I_m(x,0)} |\langle K(x,y)-K(0,y),\lambda\rangle|^2 dy\right)^{\frac{1}{2}} \\
\leq\ & \left(\int_{I_m(x,0)} \left|\sum_k 2^k \psi(2^k(x-y)) - \psi(-2^k y))h_k(2^k y)\right|^2 dy\right)^{\frac{1}{2}} \\
\leq\ & \sum_k 2^k \left(\int_{I_m(x,0)} \left|\psi(2^k(x-y)) - \psi(-2^k y))h_k(2^k y)\right|^2 dy\right)^{\frac{1}{2}} \\
=\ & \sum_k 2^{k/2} \left(\int_{I_{m+k}(x,0)} \left|\psi(2^k x-y) - \psi(-y))h_k(y)\right|^2 dy\right)^{\frac{1}{2}} \\
\leq\ & \sum_k 2^{k/2} \left(\sup_{y\in I_{m+k}(x,0)} |\psi(2^k x-y)-\psi(-y)|\right) \left(\int_{I_{m+k}(x,0)} |h_k(y)|^2 dy\right)^{\frac{1}{2}} \\
\leq\ & C\sum_k 2^{k/2} 2^{(k+m)/2} \left(\sup_{y\in I_{m+k}(x,0)} |\psi(2^k x-y)-\psi(-y)|\right) \\
=\ & C\left(\sum_{k=-\infty}^{-n-1} + \sum_{k=-n}^{\infty}\right)\left[2^{k/2}2^{(k+m)/2}\left(\sup_{y\in I_{m+k}(x,0)} |\psi(2^k x-y)-\psi(-y)|\right)\right],
\end{aligned}$$

where $n=[\frac{2m}{3}]$.

We estimate both the terms separately. For the first sum, we see that $k+m<m-n\leq m/3$, as $-\infty<k\leq -n-1$. Now we estimate using mean value theorem to get the following majorization :

$$\begin{aligned}
\sup_{y\in I_{m+k}(x,0)} |\psi(2^k x-y)-\psi(-y)| &\leq C(\psi)2^k|x| \\
&\leq C(\psi)2^{k+1}.
\end{aligned}$$

In the above, we have used that $1 \leq |x| < 2$. Hence we get

$$\begin{aligned}\sum_{k=-\infty}^{-n-1} 2^{k+\frac{m}{2}} \left(\sup_{y \in I_{m+k}(x,0)} |\psi(2^k x - y) - \psi(-y)| \right) &\lesssim \sum_{k=-\infty}^{n} 2^{k/2} 2^{(k+m)/2} 2^{k+1} \\ &\lesssim \sum_{k=-\infty}^{-n-1} 2^{3k/2} 2^{m/6} \\ &\lesssim 2^{-\frac{3}{2} \cdot \frac{2m}{3}} 2^{\frac{m}{6}} \\ &= C 2^{-\frac{5m}{6}}.\end{aligned}$$

Now we need to estimate the second sum. Note that for $y \in I_{m+k}(x,0)$, we have $2^{k+m}x \leq y < 2^{k+m+1}x$. This implies that $|2^k x - y| \simeq 2^{k+m}x$for all $y \in I_{m+k}(x,0)$ with a constant independent of y, m, and k. Also, we have that $\psi(y) \leq C|y|^{-2}$. Using these observation we get the following estimate for the second term :

$$\begin{aligned}\sup_{y \in I_{m+k}(x,0)} |\psi(2^k x - y) - \psi(-y)| &\lesssim \sup_{y \in I_{m+k}(x,0)} \left(\frac{1}{2^{2(k+m)}x} + \frac{1}{y^2} \right) \\ &\lesssim 2^{-2(k+m)}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=-n}^{\infty} 2^{k/2} 2^{(k+m)/2} \left(\sup_{y \in I_{m+k}(x,0)} |\psi(2^k x - y) - \psi(-y)| \right) &\lesssim \sum_{k=-n}^{\infty} 2^{k/2} 2^{(k+m)/2} 2^{-2(k+m)} \\ &= C \sum_{k=-n}^{\infty} 2^{-k} 2^{-3m/2} \\ &\lesssim 2^{2m/3} 2^{-3m/2} \\ &= C 2^{-\frac{5m}{6}}\end{aligned}$$

Substituting the estimates for both the terms we get,

$$\int_{I_m(x,0)} |\langle K(x,y) - K(0,y), \lambda \rangle|^2 dy \leq C 2^{-5m/6}. \tag{41}$$

This proves the required estimate (32) for the kernel $K(x,y)$ with $\alpha = \frac{5}{3} > 1$. Thus we complete the proof of Theorem 6.1. □

7. Appendix

The Radamacher functions are defined as follows :

$$r_0(t) = \begin{cases} -1, & 0 \leq t < 1/2 \\ 1, & 1/2 \leq t \leq 1, \end{cases}$$

and for $n \geq 1$, $r_n(t) = r_0(2^n t)$. The Rademacher functions form a orthonormal system in $L^2([0,1])$.

Lemma 7.1 (Khintchine's inequality). *Let $1 \leq p < \infty$. Given a sequence $\mathbf{a} = \{a_n\}$, consider the function*

$$F(t) = \sum_n a_n r_n(t).$$

Then, we have

$$\|F\|_{L^p([0.1])} \lesssim \left(\sum_n |a_n|^2\right)^{\frac{1}{2}}, \tag{42}$$

and

$$\left(\sum_n |a_n|^2\right)^{\frac{1}{2}} \lesssim \|F\|_{L^p([0.1])}, \tag{43}$$

where r_n is the Radamacher function. More importantly, the implicit constant depends only on the parameter p.

Proof. First, observe that using orthogonality property of Rademacher functions, we have

$$\|F\|_{L^2([0.1])} = \left(\sum_n |a_n|^2\right)^{\frac{1}{2}}. \tag{44}$$

Using the above estimate together with the fact that $\|.\|_{L^p([0.1])}$ increases with p, we can easily see that inequalities (42) and (43) hold for $1 \leq p \leq 2$ and $2 \leq p < \infty$ respectively. Further, we see that inequality (43) for $1 \leq p < 2$ can be deduced if we have inequality (42) for $2 < p < \infty$. For $p,\ 1 \leq p < 2$, choose $\theta \in (0,1)$ such that $\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{4}$. Then,

$$\begin{aligned}
\|F\|_{L^2([0.1])} &\leq \|F\|_{L^p([0.1])}^{1-\theta} \|F\|_{L^4([0.1])}^{\theta} \\
&\lesssim \|F\|_{L^p([0.1])}^{1-\theta} \|F\|_{L^2([0.1])}^{\theta}.
\end{aligned}$$

This proves inequality (43) for $1 \leq p < 2$. Thus, we only need to prove inequality (42) for all $2 < p < \infty$. We proceed as follows : Without loss of generality we may assume that $\|F\|_{L^2([0.1])} = 1$. Further, we observe that it is enough to prove inequality (42) for all integers bigger that 2.

We know that $|x|^p \leq p! e^{|x|} \leq p!(e^x + e^{-x}),\ x \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\|F\|_{L^p([0.1])}^p &= \int_0^1 |\sum_n a_n r_n(t)|^p dt \\
&\leq p! \int_0^1 \left(e^{\sum_n a_n r_n(t)} + e^{-\sum_n a_n r_n(t)}\right) dt \\
&= p! \left(\int_0^1 \prod_n e^{a_n r_n(t)} dt + \int_0^1 \prod_n e^{-a_n r_n(t)} dt\right) \\
&= 2p! \prod_n \int_0^1 e^{a_n r_n(t)} dt \\
&= 2p! \prod_n \frac{e^{a_n} + e^{-a_n}}{2} \\
&= 2p! \prod_n e^{a_n^2} \\
&= 2p! e.
\end{aligned}$$

Here we have used that Rademacher functions are independent variables and the inequality $\frac{e^x + e^{-x}}{2} \leq e^{x^2}$. ∎

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