SOLUTIONS AND HINTS

1. Solutions/hints

Question 1. Mark each of the following statements with true or false (no justification is required).

(1) The set of all functions $f : \mathbb{N} \to \{0, 1\}$ is countable.
(2) The set $E = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \cup \{0\}$ is closed.
(3) Let E be a set in $\mathbb{R}$. If an upper bound α of E belongs to E, the $\sup E = \alpha$.
(4) A sequence $\{a_n\}$ is convergent if and only if it is bounded.
(5) The Cauchy product of two convergent series is convergent.

Answer.

(1) False
(2) False
(3) True
(4) False
(5) False

Question 2. Short answer type questions.

(1) Give example of a bijection (one-one and onto function) between $\mathbb{N}$ and $\mathbb{Z}$.
(2) Give example of a convergent series $\sum_n a_n$ such that the series $\sum_n a_n^2$ is divergent.
(3) Find the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n!} z^n$.
(4) Let $a_n \geq 0$ and $b_n \geq 0$ be real numbers. If series $\sum_n a_n$ and $\sum_n b_n$ converge, then show that the series $\sum_n \sqrt{a_n b_n}$ converges.

Answer.

(1) The $f : \mathbb{N} \to \mathbb{Z}$ defined by
$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd} \end{cases}$$
is a one-one and onto.
(2) Take $a_n = \frac{(-1)^n}{\sqrt{n}}$ and notice that the $\sum\limits_{n=1}^{\infty} a_n$ is convergent but $\sum\limits_{n=1}^{\infty} a_n^2$ diverges.
(3) $R = +\infty$. Use ratio test.
(4) Here $a_n \geq 0$ and $b_n \geq 0$. Use that $\sqrt{a_n b_n} \leq \frac{a_n + b_n}{\sqrt{2}}$ and apply comparison test.

Question 3. Let E be a nonempty set in $\mathbb{R}$ such that it is both open and closed. Show that $E = \mathbb{R}$.
Solution. The proof is by contradiction. Suppose that E is a nonempty subset of $\mathbb{R}$ and assume that $E \neq \mathbb{R}$. Then the complement $E^c \neq \emptyset$ and $E^c \neq \mathbb{R}$. Also, $\mathbb{R} = E \cup E^c$. Take $s_0 \in E$ and $t_0 \in E^c$ be two points. Clearly, $s_0 \neq t_0$. Consider the midpoint $\frac{s_0 + t_0}{2}$, then this point is either in E or in E^c. We call this point s_1 if it belongs to E otherwise call it t_1. Continue this procedure with s_0, t_0 and the new point. This will give us two sequences s_n and t_n with $s_n \in E$ and $t_n \in E^c$. Also it is easy to verify that both the sequences converge to same point, say s. Since E and E^c are both closed sets, we see that $s \in E$ and $s \in E^c$, which is a contradiction.
Question 4. Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = a$. Show that
$$\limsup_{n\to\infty}(a_n + b_n) = a + \limsup_{n\to\infty} b_n.$$

Solution. We proved in the class that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{1}$$

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Since $\lim_{n\to\infty} a_n = a$, we get that

$$\limsup_{n\to\infty}(a_n + b_n) \leq a + \limsup_{n\to\infty} b_n$$

Using the inequality (1) we have

$$\begin{aligned}
\limsup_{n\to\infty}(b_n) &= \limsup_{n\to\infty}(a_n + b_n - a_n) \\
&\leq \limsup_{n\to\infty}(a_n + b_n) + \limsup_{n\to\infty}(-a_n) \\
&= \limsup_{n\to\infty}(a_n + b_n) - a
\end{aligned}$$

Question 5. Let $a_1 > \sqrt{2}$ be a real number. Define $a_2, a_3, a_4, \ldots$, by the recursion formula

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

Prove that $\{a_n\}$ is monotonically decreasing and $\lim_{n\to\infty} a_n = \sqrt{2}$.

Solution. Note that

$$\begin{aligned}
a_{n+1}^2 &= \frac{1}{4}(a_n + \frac{2}{a_n})^2 \\
&= \frac{1}{4}(a_n - \frac{2}{a_n})^2 + \frac{1}{4}4a_n\frac{2}{a_n} \\
&> 2.
\end{aligned}$$

Therefore, for all $n \geq 1$ we have $a_n > \sqrt{2}$. Then we also have

$$a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) < \frac{1}{2}(a_n + \frac{a_n^2}{a_n}) = a_n$$

Hence $\{a_n\}$ is monotonically decreasing and bounded from below. This implies that $\lim_{n\to\infty} a_n$ exists, call the limit l and note that $l \neq 0$. Notice that l satisfies $l = \frac{1}{2}(l + \frac{2}{l})$ and finally conclude that $l = \sqrt{2}$.

Question 6. Let $\{a_n\}$ be a monotonically decreasing sequence of non-negative real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\sum_n a_n$ is convergent then show that $\lim_{n\to\infty} na_n = 0$.

Solution. Since $\{a_n\}$ is a monotonically decreasing sequence of non-negative real numbers. Use the fact that $\sum_n a_n$ converges iff $\sum_n 2^n a_{2^n}$ converges. Hence $2^n a_{2^n} \to 0$ as $n \to \infty$.

Question 7. If $\sum_n a_n$ is a conditionally convergent series, then prove that the series of its positive terms and the series of its negative terms are both divergent.

Solution. Define $p_n = \frac{|a_n|+a_n}{2}$ and $q_n = \frac{|a_n|-a_n}{2}$ and note that p_n and q_n are non-negative with $p_n - q_n = a_n$ and $p_n + q_n = |a_n|$.

Also, note that the series of positive terms and the series of negative terms differ from $\sum p_n$ and $\sum_n q_n$ only by zero terms and therefore have similar divergence/convergence properties.

Note that both series $\sum p_n$ and $\sum_n q_n$ converge then $\sum |a_n|$ converges which contradicts the hypothesis that $\sum a_n$ is conditionally convergent. Further, if one of the series $\sum p_n$ and $\sum_n q_n$ is divergent then $\sum a_n$ can be shown to diverge, which is again a contradiction.

This completes the proof.