SOLUTIONS AND HINTS

1. Solutions of selected problems from quiz 1

Question 1. Mark each of the following statements with true or false (no justification is required).

(1) Let $\mathcal{F}(\mathbb{N})$ denote the collection of all finite subsets of $\mathbb{N}$, then $\mathcal{F}(\mathbb{N})$ is countable.
(2) The interior of $\mathbb{Q}$ is $\mathbb{Z}$.

Answer.

(1) True
(2) False

Question 2. Let $A = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$. Find $\inf A$ and $\sup A$.
Answer. $\inf A = -1$ and $\sup A = 1$.
Question 3. Show that the following statements are equivalent

(1) The Archimedian property holds.
(2) Given any $a, b \in \mathbb{R}$ with $a < b$, there exists an $r \in \mathbb{Q}$ such that $a < r < b$.
(3) Given any $a, b \in \mathbb{R}$ with $a < b$, there exist $p \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a < \frac{p}{2^n} < b$.

Solution. (1) $\Leftrightarrow$ (2) was done in the class. (3) $\Rightarrow$ (2) is easy as $\frac{p}{2^n}$ is a rational number. We prove that (1) $\Rightarrow$ (3). Suppose (1) holds and there exist $a, b \in \mathbb{R}$ with $a < b$ such that for all $p \in \mathbb{Z}$ and $n \in \mathbb{N}$ either $\frac{p}{2^n} < a$ or $b < \frac{p}{2^n}$. For each fixed $n \in \mathbb{N}$, let p_n denote the largest integer p such that $\frac{p}{2^n} < a$. From this, we get that

$$\frac{p_n}{2^n} \leq a < b \leq \frac{p_n + 1}{2^n}, \; n \in \mathbb{N}. \tag{1}$$

It is easy to verify the above inequality. For, by definition of p_n, we have $\frac{p_n}{2^n} \leq a$. Since $p_n + 1 > p_n$, it satisfies $\frac{p_n+1}{2^n} > a$. Now by our assumption we should have that $\frac{p_n+1}{2^n} > b$. Recall (1) and see that for each $n \in \mathbb{N}$, we have

$$0 < b - a \leq \frac{p_n + 1}{2^n} - \frac{p_n}{2^n} = \frac{1}{2^n}.$$

Therefore, we see that the Archimedian property fails and hence by contrapositive, we conclude that (1) $\Rightarrow$ (3).
Question 4 Prove that the set E of rational numbers in the interval $(0, 1)$ cannot be expressed as the intersection of a countable collection of open sets.
Solution. Our proof is by contradiction. Suppose that

$$E = \cap_n O_n,$$

where each O_n is an open set.
We can write $E = \{x_1, x_2, x_3, \dots\}$.
Since $x_1 \in O_1$ there exist an open interval I_1 and a closed interval J_1 such that

$$x_1 \in I_1 \subseteq O_1, \; J_1 \subseteq I_1, \text{ and } x_1 \notin J_1.$$

Note that J_1, being an interval, contains infinitely many rationals. Take a rational number and call it x_2 such that $x_2 \neq x_1$ and $x_2 \in J_1$. Since $x_2 \in O_2$, by a similar argument as above, we see that there exist an open interval I_2 and a closed interval J_2 such that

$$x_2 \in I_2 \subseteq O_2, \; J_2 \subseteq I_2 \subseteq J_1, \text{ and } x_2 \notin J_2.$$

Continuing this process inductively we get a nested sequence $\{J_n\}$ of nonempty closed and bounded intervals such that $x_n \notin J_n$, $\forall n$. Use nested interval theorem to get a contradiction.

Date: September 9, 2014.